Two-dimensional polar coding

Erdal Arıkan
Department of Electrical-Electronics Engineering
Bilkent University
Ankara, TR-06800, Turkey
Email: arikan@ee.bilkent.edu.tr

Garik Markarian
Dept. of Communication Systems
Lancaster University
Lancaster, LA1 4WA, UK
Email: g.markarian@lancaster.ac.uk

***Abstract*—Polar codes are a recently introduced class of codes that achieve the capacity of arbitrary symmetric binary-input channels. The goal of this paper is to explore a two-dimensional (2D) code based on polar coding. Unlike step-wise rate allocation in one-dimensional polar coding, the rate allocation in this 2D code is graded.**

***Index Terms*—Error correcting codes, capacity-achieving codes, polar codes, product codes.**

I. Introduction

Polar coding is a recently introduced coding technique that achieves the symmetric capacity of arbitrary binary-input discrete memoryless channels (B-DMCs) with encoding and decoding complexity bounded by $O(N \log N)$ where N is the code block-length [1]. The symmetric capacity $I(W)$ of a B-DMC W is defined as the mutual information between the channel input and output when the inputs are used with equal probability. The symmetric capacity equals the channel capacity for certain channels such as the binary erasure channel (BEC) and the binary symmetric channel (BSC).

Polar coding is based on a construction that recursively transforms N independent copies of a B-DMC W to obtain a second set of N binary-input channels $\{W_N^{(i)} : 1 \leq i \leq N\}$ such that the symmetric capacity terms $I(W_N^{(i)})$ near 0 or 1 for all but a vanishingly small fraction as N tends to infinity. The construction is applicable for any block-length $N = 2^n$ for $n \geq 0$.

The concern in [1] was mainly theoretical and the focus was mainly on asymptotic properties of polar codes. For finite block sizes, the polarization effect is not perfect and many of the channels $W_N^{(i)}$ have symmetric capacities $I(W_N^{(i)})$ that are far from being polarized as shown in Fig. 1 for the case of a BEC with erasure probability $\frac{1}{2}$. The polar coding method as presented in [1] has a step-wise rate allocation algorithm, which allocates a rate of 0 or 1 to each subchannel $W_N^{(i)}$according as $I(W_N^{(i)})$ is above a certain threshold or not. For moderate length codes, where the asymptotic effects have not yet taken hold, it is conceivable that the code performance can be improved by a graded rate allocation method so that each subchannel $W_N^{(i)}$ is assigned a rate from a larger set of possible rates. One convenient method of graded rate allocation is to consider a 2D array code, a well-known technique in coding theory [2]. The goal of this paper is to consider such a coding scheme based on polar codes.

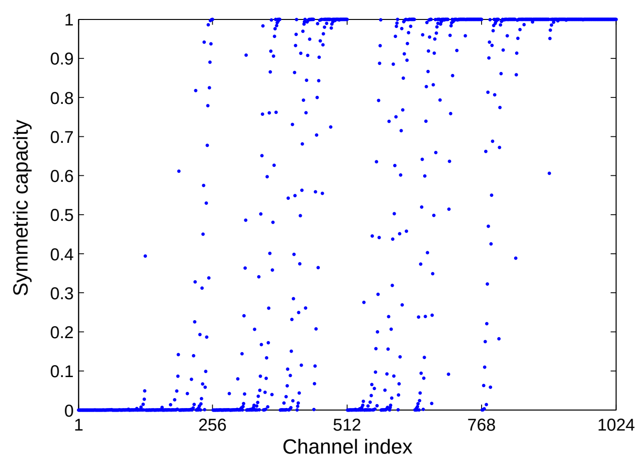

Fig. 1. Plot of $I(W_N^{(i)})$ vs. $i = 1, \ldots, N = 2^{10}$ for a BEC with erasure probability $\frac{1}{2}$.

We begin with a review of the polar code construction method.

II. Polar code construction

For any $n \geq 0$, $N = 2^n$, and $0 \leq K \leq N$, there exists a polar code with block-length N and dimension K, denoted $P(N, K)$. A polar code $P(N, K)$ is a linear code over GF(2) with a generator matrix $G_P(N, K)$ constructed in accordance with the rules described in [1]. First, an N-by-N matrix G_N is formed by the formula $G_N = B_N F^{\otimes n}$ where B_N is the *bit-reversal* operator and $F^{\otimes n}$ denotes the nth Kronecker power of F. Then, $G_P(N, K)$ is selected as a submatrix of G_N according to a selection rule defined in [1]. Although the same G_N is used for all B-DMCs, the selection rule for $G_P(N, K)$ is channel-specific. The selection rule for BECs is particularly simple. Since the goal of this paper is mainly demonstrative, we will restrict the discussion to a BEC with erasure probability $\frac{1}{2}$, unless stated otherwise.

As an example, let us consider the construction of a $P(8, 4)$

code for this case. First, we form the generator matrix

$$G_8 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&0&1&0&1&0&1&0\\ 1&1&0&0&0&0&0&0\\ 1&1&0&0&1&1&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&1&1&1&1 \end{bmatrix}.$$

Then the polar code selection rule in [1] selects the generator matrix of $P(8,4)$ as

$$G_P(8,4) = \begin{bmatrix} 1&0&1&0&1&0&1&0\\ 1&1&0&0&1&1&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&1&1&1&1 \end{bmatrix}.$$

This matrix is a submatrix of G_8. The code $P(8,4)$ maps data words $u_1^8 = (u_1, \ldots, u_8)$ to codewords $x_1^8 = (x_1, \ldots, x_8)$ via $x_1^8 = u_1^8 G_8$ where the data words u_1^8 are restricted to vectors over $GF(2)$ such that $u_1 = u_2 = u_3 = u_5 = 0$. This is equivalent to the usual viewpoint of encoding as the mapping $x_1^8 = (u_4, u_6, u_7, u_8)G_P(8,4)$. When considering the full vector u_1^8, we refer to the subvector (u_1, u_2, u_3, u_5) as the *frozen* vector and to (u_4, u_6, u_7, u_8) as the *information* vector. A vector u_1^8 an *admissible* source vector for the code $P(8,4)$ if its frozen part equals the zero vector.

III. Product coding

A 2D coding strategy with polar coding is to select an information array $U = (u_{i,j})$ such that each row of U is admissible as an information vector for a particular $P(N,K)$ code and each column is a codeword in a certain set of block codes. For example, consider the case where U is a 4-by-8 array with each row an admissible data vector for $P(8,5)$ and the columns of U selected as codewords in the length-4 codes as indicated in Table 1. An entry (N, K, d) in the table designates a code with block-length N, dimension K, and minimum distance d. For example, $(4, 0, \infty)$ designates a trivial code consisting of only the all-zero codeword. The code $(4, 1, 4)$ consists of two codewords 0000 and 1111. For

TABLE I
Column codes for the 4-by-8 code example.

Column	Code type
1	$(4, 0, \infty)$
2	$(4, 0, \infty)$
3	$(4, 0, \infty)$
4	$(4, 1, 4)$
5	$(4, 0, \infty)$
6	$(4, 3, 2)$
7	$(4, 3, 2)$
8	$(4, 4, 1)$

example, an admissible source array for this code is

$$U = \begin{bmatrix} 0&0&0&1&0&1&0&1\\ 0&0&0&1&0&0&1&1\\ 0&0&0&1&0&1&0&0\\ 0&0&0&1&0&0&1&1 \end{bmatrix}.$$

This array is converted to a codeword array $X = UG_8$ and transmitted over 32 independent copies of BEC($\frac{1}{2}$). Let Y denote the 4-by-8 received array at the channel output. Decoding is performed in interlaced row and column operations. The decoder knows that the first three columns of U are fixed to zero. So, it sets the estimates of these columns to zero: $\hat{u}_{i,j} = 0$ for $1 \leq i, j \leq 3$. The real decoding task begins with the fourth column of U, which is the first column containing information. First, independent row-by-row decoding operations are carried out to generate the estimates $\tilde{u}_{i,4}$, $1 \leq i \leq 4$, for the elements of the fourth column; this decoding step ignores the constraints imposed by the (4,1,4) code checking on the 4th column, using only the constraints imposed by the polar code. A successive cancellation (SC) decoder is suitable for this task. Next, $\tilde{u}_{i,4}$, $1 \leq i \leq 4$ are used by an ML decoder designed for the $(4, 1, 4)$ column code to generate the final decisions $\hat{u}_{i,4}$, $1 \leq i \leq 4$ for the fourth column. Next, the decoder moves on to the 5th column, which is a column frozen to zero. So, the decoder sets $\hat{u}_{i,5} = 0$ for $1 \leq i \leq 4$ and moves over to column 6. Column 6 is not a frozen column; so, a decoding operation similar to the one on column 4 is carried out. Similarly, columns 7 and 8 are decoded, and the decoding task is finished.

TABLE II
Erasure probabilities for the 4-by-8 code

	Code type			
Column index j	(4,4,1)	(4,3,2)	(4,1,4)	(4,0,∞)
1	1.0E+000	1.0E+000	9.8E-001	0
2	8.8E-001	8.8E-001	6.0E-001	0
3	8.1E-001	8.0E-001	4.3E-001	0
4	3.2E-001	2.2E-001	1.0E-002	0
5	6.8E-001	6.6E-001	2.2E-001	0
6	1.9E-001	9.0E-002	1.3E-003	0
7	1.2E-001	3.9E-002	2.2E-004	0
8	3.9E-003	4.6E-005	2.3E-010	0

The performance of this code on BEC($\frac{1}{2}$) can be estimated using Table II. This table lists the erasure probabilities for various bits of the array code. The entries under the column (4,4,1) relate to the case where effectively no column code is used; the entries in this column equal the channel parameters $Z_8^{(j)}$ obtained by the recursion

$$Z_{2k}^{(2i-1)} = 2Z_k^{(i)} - \left(Z_k^{(i)}\right)^2$$
$$Z_{2k}^{(2i)} = \left(Z_k^{(i)}\right)^2$$

for $1 \leq i \leq k$, starting with $Z_1^{(1)} = 1$. The parameter $Z_8^{(j)}$ can be interpreted as the erasure probability as seen by a SC decoder in decoding the jth bit of a polar code P(8,8) on a BEC($\frac{1}{2}$), as explained in more detail in [1].

In general we will denote an entry of this table by $Z(j, \ell)$, where j indexes columns of the 2D codeword and ℓ indexes the type of candidate column codes that may be used. E.g., $Z(5, 1) = 6.8 \cdot 10^{-1}$ and $Z(2, 3) = 6 \cdot 10^{-2}$. The entry $Z(j, \ell)$ is interpreted as the erasure probability at the decoder output for any designated bit of the ℓth code. For example, we have

$$Z(j, 2) = Z(j, 1)(1 - (1 - Z(j, 1))^3),$$

which is the probability that a designated bit of the (4,3,2) code is erased (when used on column j) and at least one other bit is also erased, so that the decoder cannot recover the erasure in the designated bit.

If we denote the parameter of the actual column code selected for use in the jth column by $(N_{\ell_j}, K_{\ell_j}, d_{\ell_j})$, then the block-error rate (BLER) P_e for the overall code is bounded as

$$P_e \leq \sum_{j=1}^{N} K_j Z(j, \ell_j).$$

The overall coding rate is given by

$$R = \frac{1}{N} \sum_{j=1}^{N} \frac{K_{\ell_j}}{N_{\ell_j}}.$$

For example, consider the case where the column codes are assigned as in Table III. The rate of the array code is then

TABLE III
AN ASSIGNMENT OF COLUMN CODES FOR THE 4-BY-8 CASE

j	$(N_{\ell_j}, K_{\ell_j}, d_{\ell_j})$	K_{ℓ_j}/N_{ℓ_j}	$Z(j, \ell_j)$
1	$(4, 0, \infty)$	0	0
2	$(4, 0, \infty)$	0	0
3	$(4, 0, \infty)$	0	0
4	$(4, 1, 4)$	0.25	1.0E-2
5	$(4, 0, \infty)$	0	0
6	$(4, 3, 2)$	0.75	9.0E-2
7	$(4, 4, 1)$	1	1.2E-1
8	$(4, 4, 1)$	1	3.9E-3

$R = 3/8$ and the BLER is bounded roughly by 2.1E-1.

Figure 1 shows simulation results for three array codes. We note that the 2D 4×256 array code achieves significantly better performance than the 1D 1×256 ordinary polar code. This proves that the effectiveness of the graded rate allocation.

It is natural to compare the performance of the 4×256 code with that of the 1×1024 polar code. This comparison shows that the two codes perform roughly the same and also have roughly the same complexity. Recall from [1] that the complexity of SC decoding of a polar code of length N is $O(N \log N)$. The decoding complexities of the codes in Table 1 are bounded by a constant c independent of N. So, a 2D code with size $4 \times N/4$ has decoding complexity approximately $O(N \log N)$ as does the $1 \times N$ polar code.

The performance may be improved by using more powerful column codes. Figure 2 gives simulation results for array codes for which the column codes available are selected from codes with parameters (8,8,1), (8,7,2), (8,4,4), (8,1,8), and (8,0,∞).

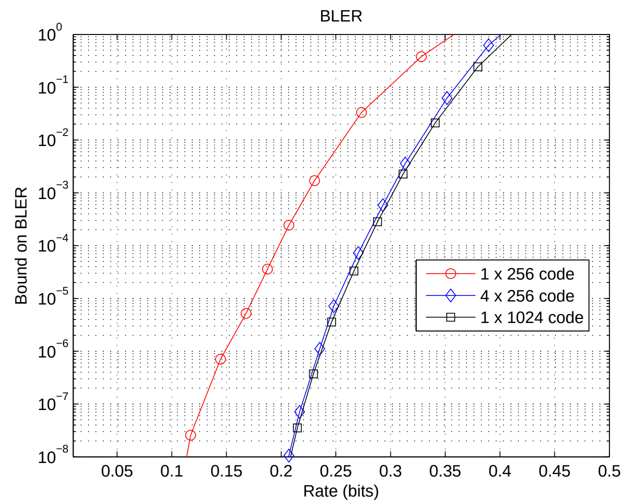

Fig. 2. Block error rates for various array codes on a BEC($\frac{1}{2}$).

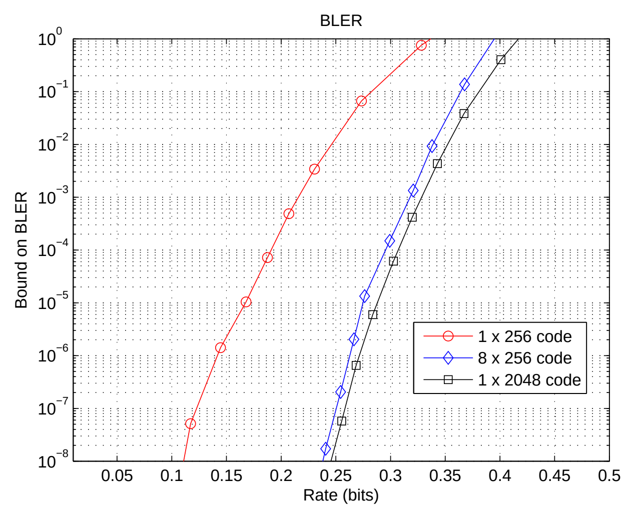

Fig. 3. Block error rates for various array codes on a BEC($\frac{1}{2}$).

In conclusion, we have shown that 2D array coding with a graded rate allocation is an effective method for improving the performance of polar coding.

ACKNOWLEDGEMENT

This work was supported in part by The Scientific and Technological Research Council of Turkey (TÜBİTAK) under project no 107E216 and in part by the European Community under FP7 NoE NEWCOM++ (grant no 216715), FP7 STREP WiMAGIC (grant no 215167), and FP7 Intra-European Fellowship Wi-CODING (grant no 219612).

REFERENCES

[1] E. Arıkan, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” 2008.
[2] B. Honary and G. Markarian, *Trellis Decoding of Block Codes.* Kluwer: Boston, 1997.