Growth Optimal Portfolios in Discrete-time Markets Under Transaction Costs

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Abstract

We investigate portfolio selection problem from a signal processing perspective and study how an investor should distribute wealth over two assets in order to maximize the cumulative wealth. We construct portfolios that provide the optimal growth in i.i.d. discrete time two-asset markets under proportional transaction costs. As the market model, we consider arbitrary discrete distributions on the price relative vectors, which can also be used to approximate a wide class of continuous distributions. To achieve optimal growth, we use threshold portfolios, where we introduce a recursive update to calculate the expected wealth. We then demonstrate that under the threshold rebalancing framework, the achievable set of portfolios elegantly form an irreducible Markov chain under mild technical conditions. We evaluate the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized using a brute force approach yielding the growth optimal portfolio under proportional transaction costs in i.i.d. discrete-time two-asset markets. As a widely known financial problem, we also solve optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets. For the case that the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator that is also incorporated in the optimization framework.

Index Terms

Growth optimal portfolio, threshold rebalancing, transaction cost, discrete-time stock market.

EDICS Category: MLR-APPL, MLR-LEAR, SSP-APPL, MLR-SLER

I. Introduction

Although analysis of financial time series and investment are extensively studied due to the amount of money involved [1]–[3], the current financial crisis demonstrated that there is a significant room for

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improvement by sound signal processing methods in financial analysis [4], [5]. Reliable signal processing methods are required in order to effectively and efficiently process the vast amount of data generated by the financial markets [3], [6]–[10]. In this paper, we investigate how and when an investor should diversify capital over various assets whose future returns are yet to be realized in order to maximize cumulative wealth. As one of the most important financial applications, this problem, i.e., the portfolio selection problem, is heavily investigated in various different fields from machine learning [11], [12], information theory [13] to signal processing [3]. In this paper, we introduce algorithms that *maximize* the expected growth in i.i.d. discrete-time markets under proportional *transaction costs*.

In particular, we consider markets that allow trading at discrete investment periods and levy transaction fees for both selling and buying proportional to the volume of trading. The market is modelled by a sequence of price relative vectors, where each entry of this vector is the ratio of the closing price to the opening price of a stock in the market. For notational simplicity we consider markets with two stocks, i.e., two-asset markets, which are extensively studied in financial literature and are shown to accurately model a wide range of financial applications [14]. The sequence of price relative vectors are assumed to have "discrete" distributions, however, the discrete distributions on the vector of price relatives are arbitrary. The corresponding discrete distributions can also be used to approximate a wide class of continuous distributions on the price relatives that satisfy certain regularity conditions by appropriately increasing the size of the discrete sample space. The detailed market model is provided in Section IV. Under this general market model, we use "threshold rebalanced portfolios" (TRP)s, which are shown to yield optimal growth in general i.i.d. discrete-time two-asset markets. We first recursively calculate the expected wealth achieved by a TRP over any investment period and then optimize the corresponding TRP to maximize expected wealth. We demonstrate that under certain technical conditions, the achievable portfolios in the TRP framework form an irreducible homogenous Markov chain with a finite number of states. This Markov chain can then be elegantly leveraged to calculate the expected growth. Subsequently, the parameters of the TRPs are optimized to achieve the maximum growth using a brute force search. Furthermore, we also solve the optimal portfolio selection problem in discrete-time markets produced by sampling continuous-time Brownian markets extensively studied in the financial literature [14].

Under mild assumptions on the sequence of price relatives and without any transaction costs, Cover et. al [13] showed that the portfolio that achieves the maximal growth is a constant rebalanced portfolio (CRP) in i.i.d. discrete time markets. A CRP is a portfolio investment strategy where the fraction of wealth invested in each stock is kept constant at each investment period. A problem extensively studied in this framework is to find sequential portfolios that asymptotically achieve the wealth of the best CRP tuned to the underlying sequence of price relatives. This amounts to finding a daily trading strategy that has the ability to perform as well as the best asset diversified, constantly rebalanced portfolio. Several sequential algorithms are introduced that achieve the performance of the best CRP either with different convergence rates or performance on historical data sets [11]–[13], [15], [16]. Even under transaction costs, sequential algorithms are introduced that achieve the performance of the best CRP [17]. Nevertheless, we emphasize that keeping a CRP may require extensive trading due to possible rebalancing at each investment period deeming CRPs ineffective in realistic markets even under mild transaction costs [18].

In continuous-time markets, however, it has been shown that under transaction costs, the optimal portfolios that achieve the maximal wealth are certain class of "no-trade zone" portfolios [19]–[21]. In simple terms, a no-trade zone portfolio has a compact closed set such that the rebalancing occurs if the current portfolio breaches this set, otherwise no rebalancing occurs. Clearly, such a no-trade zone portfolio may avoid hefty transaction costs since it can limit excessive rebalancing by defining appropriate no-trade zones. Analogous to continuous time markets, it has been shown in [22] that in two-asset i.i.d. markets under proportional transaction costs, compact no-trade zone portfolios are optimal such that they achieve the maximal growth under mild assumptions on the sequence of price relatives. In two-asset markets, the compact no trade zone is represented by thresholds, e.g., if at investment period n, the portfolio is given by $\boldsymbol{b}(n) = [b(n) \;\; (1-b(n))]^T$, where $1 \geq b(n) \geq 0$, then rebalancing occurs if $b(n) \notin (\alpha, \beta)$, given the thresholds α, β, where $1 \geq \beta \geq \alpha \geq 0$. Similarly, the interval (α, β) can be represented using a target portfolio b and a region around it, i.e., $(b-\epsilon, b+\epsilon)$, where $\min\{b, 1-b\} \geq \epsilon \geq 0$ such that $\alpha = b - \epsilon$ and $\beta = b + \epsilon$.

However, how to construct the no-trade zone portfolio, i.e., selecting the thresholds that achieve the maximal growth, has not yet been solved except in elementary scenarios [22]. We emphasize that a sequential universal algorithm that asymptotically achieves the performance of the best TRP specifically tuned to the underlying sequence of price relatives is introduced in [23]. This algorithm leverages Bayesian type weighting from [13] inspired from universal source coding and requires no statistical assumptions on the sequence of price relatives. In similar lines, various different universal sequential algorithms are introduced that achieve the performance of the best algorithm in different competition classes in [6], [7], [18], [24]–[28]. However, we emphasize that the performance guarantees in [23] (and in [6], [7], [18], [24], [26]–[28]) on the performance, although without any stochastic assumptions, is given for the worst case sequence and only optimal in the asymptotics. For any finite investment period, the corresponding order terms in the upper bounds may not be negligible in financial markets, although they may be neglected in source coding applications (where these algorithms are inspired from). We demonstrate that our algorithm readily outperforms these universal algorithms over historical data [13], where similar observations are reported in [25], [29].

Our main contributions are as follows. We first recursively evaluate the expected achieved wealth of a threshold portfolio for any b and ϵ over any investment period. We then demonstrate that under the threshold rebalancing framework, the achievable set of portfolios form an irreducible Markov chain under mild technical conditions. We evaluate the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized using a brute force approach yielding the growth optimal investment portfolio under proportional transaction costs in i.i.d. discrete-time two-asset markets. As a well studied problem, we also solve optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets. For the case that the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator to estimate the corresponding distributions that is incorporated in the optimization framework in the Simulations section.

The organization of the paper is as follows. In Section II, we briefly describe our discrete time stock market model that has two stocks with discrete price relatives and symmetric proportional transaction costs. In Section III, we start to investigate threshold rebalancing portfolios, where we first introduce a recursive update in Section III-A. We then show that the TRP framework can be analyzed using finite state Markov chains in Section III-B and Section III-C. The special Brownian market is analyzed in Section III-D. The maximum likelihood estimator is derived in Section IV. We simulate the performance of our algorithms in Section V and the paper concludes with certain remarks in Section VI.

II. Problem Description

We consider discrete-time stock markets under transaction costs. We model the market as a sequence of price relative vectors $\mathbf{X}(n)$ and consider a market with two stocks. A vector of price relatives $\mathbf{X}(n) = [X_1(n)\; X_2(n)]^T$ represents the change in the prices of the assets over investment period n, i.e., $X_i(n)$ is the ratio of the closing to the opening price of the ith stock over period n. We assume that the price relative sequences $X_1(n)$ and $X_2(n)$ are independent and identically distributed (i.i.d.) over with possibly different discrete sample spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e., $X_1(n) \in \mathcal{X}_1$ and $X_2(n) \in \mathcal{X}_2$, respectively [22]. For technical reasons, in our derivations, we assume that the sample space is $\mathcal{X} \triangleq \mathcal{X}_1 \cup \mathcal{X}_2 = \{x_1, x_2, \ldots, x_K\}$ for both $X_1(n)$ and $X_2(n)$ where $|\mathcal{X}| = K$ is the cardinality of the set $\mathcal{X}$. The probability mass function (pmf) of $X_1(n)$ is $p_1(x) \triangleq \mathrm{P}(X_1 = x)$ and the probability mass function of $X_2(n)$ is $p_2(x) \triangleq \mathrm{P}(X_2 = x)$. We define $p_{i,1} = p_1(x_i)$ and $p_{i,2} = p_2(x_i)$ for $x_i \in \mathcal{X}$ and the probability mass vectors $\mathbf{p}_1 = [p_{1,1}\; p_{2,1}\; \ldots\; p_{K,1}]^T$ and $\mathbf{p}_2 = [p_{1,2}\; p_{2,2}\; \ldots\; p_{K,2}]^T$, respectively. Here, we first assume that the corresponding probability mass vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are known. We then extend our analysis where $\mathbf{p}_1$ and $\mathbf{p}_2$ are unknown and sequentially estimated using a maximum likelihood estimator in Section IV.

An allocation of wealth over two stocks is represented by the portfolio vector $\mathbf{b}(n) = [b(n)\ 1 - b(n)]$, where $b(n)$ and $1 - b(n)$ represents the proportion of wealth invested in the first and second stocks, respectively, for each investment period n. In two stock markets, the portfolio vector $\mathbf{b} = [b\ 1 - b]$ is completely characterized by the proportion b of the total wealth invested in the first stock. For notational clarity, we use $b(n)$ to represent $b_1(n)$ throughout the paper.

We denote a threshold rebalancing portfolio with an initial and target portfolio b and a threshold ϵ by TRP(b,ϵ). At each market period n, an investor rebalances the asset allocation only if the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$. When $b(n) \notin (b-\epsilon, b+\epsilon)$, the investor buys and sells stocks so that the asset allocation is rebalanced to the initial allocation, i.e., $b(n) = b$, and he/she has to pay transaction fees. We emphasize that the rebalancing can be made directly to the closest boundry instead of to b as suggested in [22], however, we rebalance to b for notational simplicity and our derivations hold for that case also. We model transaction cost paid when rebalancing the asset allocation by a fixed proportional cost $c \in (0, 1)$ [17], [18], [22]. For instance, if the investor buys or sells S dollars of stocks, then he/she pays cS dollars of transaction fees. Although we assume a symmetric transaction cost ratio, all the results can be carried over to markets with asymmetric costs [18], [22]. Let $S(N)$ denote the achieved wealth at investment period N and assume, without loss of generality, that the initial wealth of the investor is 1 dollars. For example, if the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ and the allocation of wealth is not rebalanced for N investment periods, then the current proportion of wealth invested in the first stock is given by

$$b(N) = \frac{b\prod_{n=1}^{N} X_1(n)}{b\prod_{n=1}^{N} X_1(n) + (1-b)\prod_{n=1}^{N} X_2(n)}$$

and achieved wealth is given by

$$S(N) = b\prod_{n=1}^{N} X_1(n) + (1-b)\prod_{n=1}^{N} X_2(n).$$

If the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ at period N, i.e., $b(N) \notin (b-\epsilon, b+\epsilon)$, then the investor rebalances the asset distribution to the initial distribution and pays approximately $S(N)|b(N)-b|c$ dollars for transaction costs [17].

In the next section, we first evaluate the expected achieved wealth $E[S(N)]$ so that we can optimize b and ϵ. We then present conditions under which the set of all achievable portfolios has finite elements and derive the expected achieved wealth under these conditions. Finally, we consider the well-known Brownian market with two stocks and find the expected wealth growth [19], [22] which is then optimized.

III. Threshold Rebalanced Portfolios

In this section, we investigate threshold rebalancing portfolios in discrete-time two-asset markets under proportional transaction costs. We first calculate the expected achieved wealth at a given investment period by an iterative algorithm. Then, we present an upper bound on the complexity of the algorithm. We next give the necessary and sufficient conditions such that the achievable portfolios are finite at any investment period. This result is important when we calculate the expected achieved wealth since the complexity of the algorithm does not grow when the set of achievable portfolios is finite at any period. We also show that the portfolio sequence converges to a stationary distribution and derive the expected achieved wealth. Based on the calculation of the expected achieved wealth, we optimize b and ϵ using a brute-force search. Finally, with these derivations, we consider the well-known discrete-time two-asset Brownian market with proportional transaction costs and investigate the asymptotic expected achieved wealth to optimize b and ϵ.

A. An Iterative Algorithm

In this section, we calculate the expected wealth growth of a TRP with an iterative algorithm and find an upper bound on the complexity of the algorithm. To accomplish this, we first define the set of achievable portfolios at each investment period since the iterative calculation of the expected achieved wealth is based on the achievable portfolio set. We next introduce the portfolio transition sets and the transition probabilities of achievable portfolios at successive investment periods in order to find the probability of each portfolio state iteratively. We evaluate the expected achieved wealth $E[S(N)]$ at a given investment period N based on the set of achievable portfolios, the transition probabilities and the set of price relative vectors connecting the portfolio states. We then optimize b and ϵ using a brute-force search.

We define the set of achievable portfolios at each investment period as follows. Since the sample space of the price relative sequences $X_1(n)$ and $X_2(n)$ is finite, i.e., $|\mathcal{X}| = K$, the set of achievable portfolios at period N can only have finitely many elements. We define the set of achievable portfolios at period N as $\mathcal{B}_N = \{b_{1,N}, \ldots, b_{M_N,N}\}$, where $M_N \stackrel{\triangle}{=} |B_N|$ is the size of the set $\mathcal{B}_N$ for $N \geq 1$. As an example, we have

$$\mathcal{B}_1 = \left\{ b_{1,1}, \ldots, b_{M_1,1} \mid b_{l,1} = \frac{bu}{bu + (1-b)v} \in (b-\epsilon, b+\epsilon) \text{ or } b_{l,1} = b,\ u, v \in \mathcal{X} \right\}.$$

As illustrated in Fig. 1, for each achievable portfolio $b_{l,N} \in \mathcal{B}_N$, there is a certain set of portfolios in $\mathcal{B}_{N-1}$ that are connected to $b_{l,n}$, by definition of $b_{l,n}$. At a given investment period N, the set of achievable portfolios $\mathcal{B}_N$ is given by

$$\mathcal{B}_N = \left\{ b_{1,N}, \ldots, b_{M_N,N} \mid b_{l,N} = \frac{b_{k,N-1}u}{b_{k,N-1}u + (1-b_{k,N-1})v} \in (b-\epsilon, b+\epsilon) \text{ or } b_{l,N} = b,\ u, v \in \mathcal{X} \right\}.$$

We let, without loss of generality, $b_{1,N} = b$ for each $N \in \mathbb{N}$. Note that in Fig. 1, the size of the set of achievable portfolios at each period may grow in the next period depending on the set of price relative

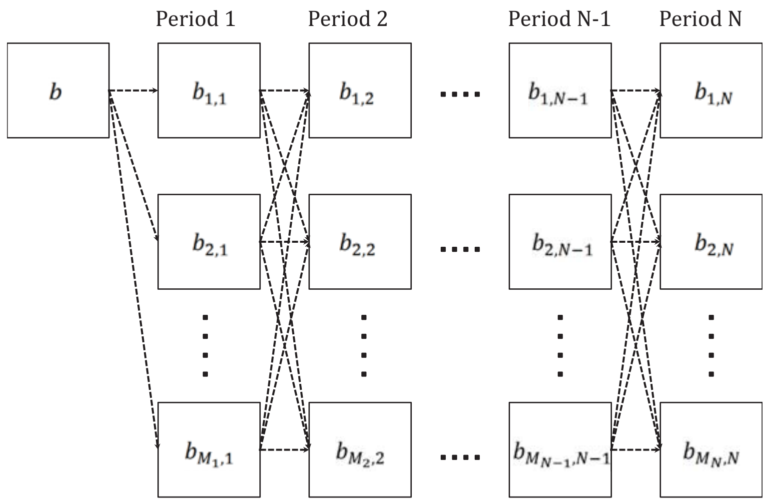


Fig. 1: Block diagram representation of N period investment.

vectors. We next define the transition probabilities as $q_{k,l,N} = \mathrm{P}\left(b(N) = b_{l,N}|b(N-1) = b_{k,N-1}\right)$ for $k = 1, \ldots, M_{N-1}$ and $l = 1, \ldots, M_N$ and the set of achievable portfolios that are connected to $b_{l,N}$, i.e., the portfolio transition set, as $\mathcal{N}_{l,N} = \{b_{k,N-1} \in \mathcal{B}_{N-1} \mid q_{k,l,N} > 0,\ \ k = 1, \ldots, M_{N-1}\}$ for $l = 1, \ldots, M_N$. Hence, the probability of each portfolio state is given by

$$\begin{aligned}\mathrm{P}\left(b(N) = b_{l,N}\right) &= \sum_{b_{k,N-1} \in \mathcal{B}_{N-1}} \mathrm{P}\left(b(N) = b_{l,N}|b(N-1) = b_{k,N-1}\right) \mathrm{P}\left(b(N-1) = b_{k,N-1}\right) \\ &= \sum_{b_{k,N-1} \in \mathcal{N}_{l,N}} q_{k,l,N} \mathrm{P}\left(b(N-1) = b_{k,N-1}\right) \end{aligned} \tag{1}$$

for $l = 1, \ldots, M_N$. Therefore, we can calculate the probability of achievable portfolios iteratively. Using these iterative equations, we next iteratively calculate the expected achieved wealth $E[S(N)]$ at each period as follows.

By definition of $\mathcal{B}_N$ and using the law of total expectation [30], the expected achieved wealth at investment period N can be written as

$$E[S(N)] = \sum_{l=1}^{M_N} \mathrm{P}\left(b(N) = b_{l,N}\right) E\left[S(N)|b(N) = b_{l,N}\right]. \tag{2}$$

To get $E[S(N)]$ in (2) iteratively, we evaluate $\mathrm{P}\left(b(N) = b_{l,N}\right) E\left[S(N)|b(N) = b_{l,N}\right]$ for each $l = 1, \ldots, M_N$ from $\mathrm{P}\left(b(N-1) = b_{k,N-1}\right) E[S(N-1)|b(N-1) = b_{k,N-1}]$ for $k = 1, \ldots, M_{N-1}$. To achieve this, we first find the transition probabilities (not the state probabilities) between the achievable portfolios.

We define the set of price relative vectors that connect $b_{k,N-1}$ to $b_{l,N}$ as $\mathcal{U}_{k,l,N}$ where

$$\mathcal{U}_{k,l,N} = \left\{ \mathbf{w} = [w_1 \ w_2]^T \in \mathcal{X}^2 \mid b_{l,N} = \frac{w_1 b_{k,N-1}}{w_1 b_{k,N-1} + w_2(1 - b_{k,N-1})} \right\}$$

for $k = 1, \ldots, M_{N-1}$ and $l = 2, \ldots, M_N$. We consider the price relative vectors that connect $b_{k,N-1}$ to $b_{1,N} = b$ separately since, in this case, there are two cases depending on whether the portfolio leaves the interval $(b - \epsilon, b + \epsilon)$ or not. We define $\mathcal{U}_{k,1,N}$ as

$$\mathcal{U}_{k,1,N} = \mathcal{V}_{k,1,N} \cup \mathcal{R}_{k,1,N},$$

where $\mathcal{V}_{k,1,N}$ is the set of price relative vectors that connect $b_{k,N-1}$ to $b_{1,N} = b$ such that the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$ at period N, i.e.,

$$\mathcal{V}_{k,1,N} = \left\{ \mathbf{w} = [w_1 \ w_2]^T \in \mathcal{X}^2 \mid \frac{w_1 b_{k,N-1}}{w_1 b_{k,N-1} + w_2(1 - b_{k,N-1})} = b \right\},$$

and $\mathcal{R}_{k,1,N}$ is the set of price relative vectors that connect $b_{k,N-1}$ to $b_{1,N}$ such that the portfolio leaves the interval $(b - \epsilon, b + \epsilon)$ at period N and is rebalanced to $b_{1,N} = b$, i.e.,

$$\mathcal{R}_{k,1,N} = \left\{ \mathbf{w} = [w_1 \ w_2]^T \in \mathcal{X}^2 \mid \frac{w_1 b_{k,N-1}}{w_1 b_{k,N-1} + w_2(1 - b_{k,N-1})} \notin (b - \epsilon, b + \epsilon) \right\}.$$

Then, the transition probabilities are given by

$$\begin{aligned} q_{k,l,N} &= \mathrm{P}\left(b(N) = b_{l,N} | b(N-1) = b_{k,N-1}\right) = \mathrm{P}\left(\mathbf{X}(N) \in \mathcal{U}_{k,l,N}\right) \\ &= \sum_{\mathbf{w} = [w_1 \ w_2]^T \in \mathcal{U}_{k,l,N}} p_1(w_1) p_2(w_2) \end{aligned} \tag{3}$$

for $k = 1, \ldots, M_{N-1}$ and $l = 1, \ldots, M_N$ so that we can calculate $\mathrm{P}\left(b(N)\right) = b_{l,N})$ iteratively for each $l = 1, \ldots, M_N$ by (1). Since we have recursive equations for the state probabilities, we next perform the iterative calculation of the expected achieved wealth based on the achievable portfolio sets and the transition probabilities.

Given the recursive formulation for the state probabilities, we can evaluate the term $\mathrm{P}\left(b(N) = b_{l,N}\right) E[S(N) | b(N) = b_{l,N}]$ for $l = 1, \ldots, M_N$ from $\mathrm{P}\left(b(N-1) = b_{k,N-1}\right) E[S(N-1) | b(N-1) = b_{k,N-1}]$ for $k = 1, \ldots, M_{N-1}$ iteratively to calculate $E[S(N)]$ by (2) as follows. To evaluate $\mathrm{P}\left(b(N) = b_{l,N}\right) E[S(N) | b(N) = b_{l,N}]$, we need to consider two cases separately based on the value of $b_{l,N}$.

In the first case, we see that if the portfolio $b(N) = b_{l,N}$, where $l = 2, \ldots, N$, then the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$ at period N. Hence, no transaction cost is paid so that we can express $\mathrm{P}\left(b(N) = b_{l,N}\right) E[S(N) | b(N) = b_{l,N}]$ as a summation of the conditional expectations for all

$b_{k,N-1} \in \mathcal{N}_{l,N}$ by the law of total expectation [30] as

$$
\begin{aligned}
&\mathrm{P}\left(b(N)=b_{l,N}\right) E[S(N)|b(N)=b_{l,N}] \\
&= \sum_{b_{k,N-1}\in\mathcal{N}_{l,N}} E\left[S(N)|b(N)=b_{l,N}, b(N-1)=b_{k,N-1}\right] \mathrm{P}\left(b(N-1)=b_{k,N-1}|b(N)=b_{l,N}\right) \mathrm{P}\left(b(N)=b_{l,N}\right) \\
&= \sum_{b_{k,N-1}\in\mathcal{N}_{l,N}} E\left[S(N)|b(N)=b_{l,N}, b(N-1)=b_{k,N-1}\right] \mathrm{P}\left(b(N-1)=b_{k,N-1}\right) q_{k,l,N}, \qquad (4)
\end{aligned}
$$

where (4) follows from Bayes' theorem [31]. We note that given $b(N-1) = b_{k,N-1}$ and $b(N) = b_{l,N}$, the price relative vector $\mathbf{X}(N)$ can take values from $\mathcal{U}_{k,l,N}$ and $q_{k,l,N} = \mathrm{P}\left(\mathbf{X}(N) \in \mathcal{U}_{k,l,N}\right)$ so that (4) can be written as a summation of the conditional expectations for all $\mathbf{X}(N) = \mathbf{w} \in \mathcal{U}_{k,l,N}$ [30] after replacing $q_{k,l,N}$

$$
\begin{aligned}
&\mathrm{P}\left(b(N)=b_{l,N}\right) E[S(N)|b(N)=b_{l,N}] \\
&= \sum_{b_{k,N-1}\in\mathcal{N}_{l,N}} \sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{U}_{k,l,N}} E\left[S_N|b(N)=b_{l,N}, b(N-1)=b_{k,N-1}, \mathbf{X}(N)=\mathbf{w}\right] \\
&\times \mathrm{P}\left(b(N-1)=b_{k,N-1}\right) \mathrm{P}\left(\mathbf{X}(N)=\mathbf{w}|\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right) \mathrm{P}\left(\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right). \qquad (5)
\end{aligned}
$$

Now, given that $b(N-1) = b_{k,N-1}$, $b(N) = b_{l,N}$ and $\mathbf{X}(N) = \mathbf{w} = [w_1\ w_2]^T$, we observe that $\mathrm{P}\left(\mathbf{X}(N)=\mathbf{w}|\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right)\mathrm{P}\left(\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right) = \mathrm{P}\left(\mathbf{X}(N)=\mathbf{w}\right)$ and

$$
\begin{aligned}
&E\left[S_N|b(N)=b_{l,N}, b(N-1)=b_{k,N-1}, \mathbf{X}(N)=\mathbf{w}\right] \\
&= E\left[S(N-1)(b_{k,N-1}w_1 + (1-b_{k,N-1})w_2)|b(N-1)=b_{k,N-1}\right], \qquad (6)
\end{aligned}
$$

and by using (6) in (5), we have

$$
\begin{aligned}
&\mathrm{P}\left(b(N)=b_{l,N}\right) E[S(N)|b(N)=b_{l,N}] \\
&= \sum_{b_{k,N-1}\in\mathcal{N}_{l,N}} \sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{U}_{k,l,N}} E\left[S(N-1)(b_{k,N-1}w_1 + (1-b_{k,N-1})w_2)|b(N-1)=b_{k,N-1}\right] \\
&\times \mathrm{P}\left(b(N-1)=b_{k,N-1}\right) \mathrm{P}\left(\mathbf{X}(N)=\mathbf{w}\right).
\end{aligned}
$$

Therefore, we can write $\mathrm{P}\left(b(N)=b_{l,N}\right) E[S(N)|b(N)=b_{l,N}]$ from $\mathrm{P}\left(b(N-1)=b_{k,N-1}\right) E[S(N-1)|b(N-1)=b_{k,N-1}]$ as

$$
\begin{aligned}
&\mathrm{P}\left(b(N)=b_{l,N}\right) E[S(N)|b(N)=b_{l,N}] \\
&= \sum_{b_{k,N-1}\in\mathcal{N}_{l,N}} \mathrm{P}\left(b(N-1)=b_{k,N-1}\right) E\left[S(N-1)|b(N-1)=b_{k,N-1}\right] \\
&\times \sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{U}_{k,l,N}} (b_{k,N-1}w_1 + (1-b_{k,N-1})w_2)p_1(w_1)p_2(w_2) \qquad (7)
\end{aligned}
$$

for $l = 2, \ldots, M_N$, where we use $\mathrm{P}\left(\mathbf{X}(N) = \mathbf{w}\right) = p_1(w_1)p_2(w_2)$.

In the second case, if the portfolio $b(N) = b_{1,N}$, then there are two sets of price relative vectors that connect $b_{k,N-1}$ to $b_{1,N}$, i.e., $\mathcal{V}_{k,1,N}$ and $\mathcal{R}_{k,1,N}$. Depending on the value of the price vector, the portfolio may be rebalanced to $b_{1,N} = b$. If $\mathbf{X}(N) \in \mathcal{V}_{k,1,N}$, then the portfolio is not rebalanced and no transaction fee is paid. If $\mathbf{X}(N) \in \mathcal{R}_{k,1,N}$, then the portfolio is rebalanced and transaction cost is paid. We can find $\mathrm{P}\left(b(N) = b_{1,N}\right) E[S(N)|b(N) = b_{1,N}]$ from $\mathrm{P}\left(b(N-1) = b_{k,N-1}\right) E[S(N-1)|b(N-1) = b_{k,N-1}]$ as a summation of the conditional expectations for all $b_{k,N-1} \in \mathcal{N}_{1,N}$ [30] as

$$
\begin{aligned}
&\mathrm{P}\left(b(N) = b_{1,N}\right) E[S(N)|b(N) = b_{1,N}] \\
&= \sum_{b_{k,N-1} \in \mathcal{N}_{1,N}} E\left[S(N)|b(N) = b_{1,N}, b(N-1) = b_{k,N-1}\right] \mathrm{P}\left(b(N-1) = b_{k,N-1}|b(N) = b_{1,N}\right) \mathrm{P}\left(b(N) = b_{1,N}\right) \\
&= \sum_{b_{k,N-1} \in \mathcal{N}_{1,N}} E\left[S(N)|b(N) = b_{1,N}, b(N-1) = b_{k,N-1}\right] \mathrm{P}\left(b(N-1) = b_{k,N-1}\right) q_{k,l,N}. \qquad (8)
\end{aligned}
$$

We note that given $b(N-1) = b_{k,N-1}$ and $b(N) = b_{1,N}$, the price relative vector $\mathbf{X}(N)$ can take values from $\mathcal{V}_{k,1,N}$ or $\mathcal{R}_{k,1,N}$, $q_{k,l,N} = \mathrm{P}\left(\mathbf{X}(N) \in \mathcal{U}_{k,l,N}\right)$ and $\mathrm{P}\left(\mathbf{X}(N) = \mathbf{w}|\mathbf{X}(N) \in \mathcal{U}_{k,l,N}\right) \mathrm{P}\left(\mathbf{X}(N) \in \mathcal{U}_{k,l,N}\right) = \mathrm{P}\left(\mathbf{X}(N) = \mathbf{w}\right)$ which yields in (8) that

$$
\begin{aligned}
&\mathrm{P}\left(b(N) = b_{1,N}\right) E[S(N)|b(N) = b_{1,N}] \\
&= \sum_{b_{k,N-1} \in \mathcal{N}_{l,N}} \left\{ \sum_{\mathbf{w}=[w_1\ w_2]^T \in \mathcal{V}_{k,1,N}} E\left[S_N|b(N) = b_{l,N}, b(N-1) = b_{k,N-1}, \mathbf{X}(N) = \mathbf{w}\right] \right. \\
&\times \mathrm{P}\left(b(N-1) = b_{k,N-1}\right) \mathrm{P}\left(\mathbf{X}(N) = \mathbf{w}\right) \\
&\left. + \sum_{\mathbf{w}=[w_1\ w_2]^T \in \mathcal{R}_{k,1,N}} E\left[S_N|b(N) = b_{l,N}, b(N-1) = b_{k,N-1}, \mathbf{X}(N) = \mathbf{w}\right] \mathrm{P}\left(b(N-1) = b_{k,N-1}\right) \mathrm{P}\left(\mathbf{X}(N) = \mathbf{w}\right) \right\}.
\end{aligned}
$$

If $\mathbf{X}(N) = \mathbf{w} \in \mathcal{V}_{k,1,N}$, then it follows that

$$
\begin{aligned}
&E\left[S_N|b(N) = b_{1,N}, b(N-1) = b_{k,N-1}, \mathbf{X}(N) = \mathbf{w}\right] \\
&= E\left[S(N-1)(b_{k,N-1}w_1 + (1 - b_{k,N-1})w_2)|b(N-1) = b_{k,N-1}\right]. \qquad (9)
\end{aligned}
$$

If $\mathbf{X}(N) = \mathbf{w} \in \mathcal{R}_{k,1,N}$, then transaction cost is paid which results

$$
\begin{aligned}
&E\left[S_N|b(N) = b_{1,N}, b(N-1) = b_{k,N-1}, \mathbf{X}(N) = \mathbf{w}\right] \\
&= E\left[S(N-1)(b_{k,N-1}w_1 + (1 - b_{k,N-1}))\left(1 - c\left|\frac{b_{k,N-1}w_1}{b_{k,N-1}w_1 + (1 - b_{k,N-1})w_2} - b\right|\right)\middle|b(N-1) = b_{k,N-1}\right]. \\
&\qquad (10)
\end{aligned}
$$

Hence, we can write (8) after using (9) and (10) as

$$
\begin{aligned}
&\mathrm{P}\left(b(N)=b_{1,N}\right)E[S(N)|b(N)=b_{1,N}] \\
&=\sum_{b_{k,N-1}\in\mathcal{N}_{1,N}}\mathrm{P}\left(b(N-1)=b_{k,N-1}\right) \\
&\times\Bigg\{\sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{V}_{k,1,N}}\mathrm{P}\left(\mathbf{X}(N)=\mathbf{w}\right)E\left[S(N-1)(b_{k,N-1}w_1+(1-b_{k,N-1})w_2)|b(N-1)=b_{k,N-1}\right] \\
&+\sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{R}_{k,1,N}}\mathrm{P}\left(\mathbf{X}(N)=\mathbf{w}\right) \qquad (11)\\
&\times E\left[S(N-1)(b_{k,N-1}w_1+(1-b_{k,N-1}))\left(1-c\left|\frac{b_{k,N-1}w_1}{b_{k,N-1}w_1+(1-b_{k,N-1})w_2}-b\right|\right)|b(N-1)=b_{k,N-1}\right]\Bigg\}.
\end{aligned}
$$

Thus, we can write $\mathrm{P}\left(b(N)=b_{1,N}\right)E[S(N)|b(N)\ =\ b_{1,N}]$ from $\mathrm{P}\left(b(N-1)=b_{k,N-1}\right)E[S(N-1)|b(N-1)=b_{k,N-1}]$ as

$$
\begin{aligned}
&\mathrm{P}\left(b(N)=b_{1,N}\right)E[S(N)|b(N)=b_{1,N}] \\
&=\sum_{b_{k,N-1}\in\mathcal{N}_{1,N}}\mathrm{P}\left(b(N-1)=b_{k,N-1}\right)E\left[S(N-1)|b(N-1)=b_{k,N-1}\right] \\
&\times\Bigg\{\sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{V}_{k,1,N}}(b_{k,N-1}w_1+(1-b_{k,N-1})w_2)p_1(w_1)p_2(w_2) \qquad (12)\\
&+\sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{R}_{k,1,N}}(b_{k,N-1}w_1+(1-b_{k,N-1}))\left(1-c\left|\frac{b_{k,N-1}w_1}{b_{k,N-1}w_1+(1-b_{k,N-1})w_2}-b\right|\right)p_1(w_1)p_2(w_2)\Bigg\},
\end{aligned}
$$

which yields the recursive expressions for $\mathrm{P}\left(b(N)=b_{l,N}\right)E[S(N)|b(N)\ =\ b_{l,N}]$ iteratively for each $l=1,\ldots,M_N$ with (7) and (12).

Hence, in the first case where the portfolio $b(N)\ =\ b_{l,N}$ for $l\ =\ 2,\ldots M_N$, we can calculate $E\left[S(N)|b(N)=b_{l,N}\right]\mathrm{P}\left(b(N)=b_{l,N}\right)$ from $E\left[S(N-1)|b(N-1)=b_{k,N-1}\right]\mathrm{P}\left(b(N-1)=b_{k,N-1}\right)$ for $b_{k,N-1}\in\mathcal{N}_{l,N}$ by (7). In the second case where the portfolio $b(N)=b_{1,N}=b$, we can calculate $E\left[S_N|b(N)=b_{1,N}\right]\mathrm{P}\left(b(N)=b_{1,N}\right)$ from $E\left[S(N-1)|b(N-1)=b_{k,N-1}\right]\mathrm{P}\left(b(N-1)=b_{k,N-1}\right)$ for $b_{k,N-1}\in\mathcal{N}_{1,N}$ by (12). Therefore, we can evaluate $E[S(N)]$ iteratively by (2). Since, we have the recursive formulation, we can optimize b and ϵ by a brute force search as shown in the Simulations section. For this recursive evaluation, we have to find the set of achievable portfolios at each investment period to compute $E[S(N)]$ by (2). Hence, we next analyze the number of calculations required to evaluate the expected achieved wealth $E[S(N)]$.

In the following lemma, we investigate the number of achievable portfolios at a given market period to determine the complexity of the iterative algorithm. We show that the set of achievable portfolios at period N is equivalent to the set of achievable portfolios when the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods. We first demonstrate that if the portfolio never leaves the interval $(b-\epsilon, b+\epsilon)$ for N periods, then $b(N)$ is given by

$$b(N) = \frac{1}{1 + \frac{1-b}{b} e^{\sum_{n=1}^{N} Z(n)}},$$

where $Z(n) \overset{\triangle}{=} \ln \frac{X_2(n)}{X_1(n)}$ with a sample space $\mathcal{Z} = \left\{ z = \ln \frac{u}{v} \mid u, v \in \mathcal{X} \right\}$ where $|\mathcal{Z}| = M$. Then, we argue that the number of achievable portfolios at period N, M_N, is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take when the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods. We point out that $M \leq K^2 - K + 1$ since the price relative sequences $X_1(n)$ and $X_2(n)$ are elements of the same sample space $\mathcal{X}$ with $|\mathcal{X}| = K$ and by using this, we find an upper bound on the number of achievable portfolios.

Lemma 3.1: The number of achievable portfolios at period N, M_N, is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take when the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods and is bounded by $\binom{N+K^2-K}{N}$, i.e., $M_N = |\mathcal{B}_N| \leq \binom{N+K^2-K}{N}$.

Proof: The proof is in the Appendix A. ■

Hence, the complexity of calculating $E[S(N)]$ is bounded by $\mathcal{O}\left(\sum_{n=1}^{N} \binom{n+K^2-K}{n} / N\right)$ since at each period $n = 1, \ldots, N$, we calculate $E[S(n)]$ as a summation of M_n terms, i.e., $E[S(n)] = \sum_{l=1}^{M_n} E[S(n) | b(n) = b_{l,n}] \mathrm{P}\left(b(n) = b_{l,n}\right)$ and $M_n \leq \binom{n+K^2-K}{n}$.

In the next section, we show that the set of all achievable portfolios, $\mathcal{B} \overset{\triangle}{=} \cup_{n=1}^{\infty} \mathcal{B}_n$, is finite under mild technical conditions. This result is important when we analyze the asymptotic behavior of the expected achieved wealth since the the complexity of the algorithm that evaluates $E[S(n)]$ is constant when the set of achievable portfolios is finite. We demonstrate that the portfolio sequence forms a Markov chain with a finite state space and converges to a stationary distribution. Finally, we analyze the limiting behavior of the expected achieved wealth and then optimize b and ϵ with a brute-force algorithm.

B. Finitely Many Achievable Portfolios

In this section, we investigate the cardinality of the set of achievable portfolios $\mathcal{B}$ and demonstrate that $\mathcal{B}$ is finite under certain conditions in the following theorem, Theorem 3.1. This result is significant since when $\mathcal{B}$ is finite, we can derive a recursive update with a constant complexity, i.e., the number of states does not grow, to calculate the expected achieved wealth at any investment period. Then, we can investigate the limiting behavior of the expected achieved wealth using this update to optimize b and ϵ.

Before providing the main theorem, we first state a couple of lemmas that are used in the derivation of the main result of this section.

We first point out that in Lemma 3.1, we showed that the number of achievable portfolios at period N is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take when the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods. Then, we observed that the cardinality of the set $\mathcal{B}$ is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ when the portfolio $b(n)$ never leaves the interval $(b-\epsilon, b+\epsilon)$. We next show that the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N periods if and only if the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$, where $\alpha_1 \triangleq \ln \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)} > 0$ and $\alpha_2 \triangleq \ln \frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} < 0$. Moreover, we also prove that the number of achievable portfolios is equal to the cardinality of the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ where we define the set $\mathcal{M}$ as

$$\mathcal{M} = \{m_1 z_1 + m_2 z_2 + \ldots + m_{M^+} z_{M^+} \mid m_i \in \mathbb{Z},\ z_i \in \mathcal{Z}^+ \text{ for } i = 1, \ldots, M^+\}, \tag{13}$$

$\mathcal{Z}^+ \triangleq \{z \in \mathcal{Z} \mid z \geq 0\}$, $M^+ \triangleq |\mathcal{Z}^+|$. Note that $\mathcal{Z}^+$ is the set of positive elements of the set $\mathcal{Z}$ and any value that the sum $\sum_{n=1}^{N} Z(n)$ can take is an element of $\mathcal{M}$. Hence, if we can demonstrate that the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is finite under certain conditions, then it yields the cardinality of the set $\mathcal{B}$ since $\mathcal{B}$ is finite if and only if $M \cap (\alpha_2, \alpha_1)$ is finite.

In the following lemma, we prove that the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N periods if and only if the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$.

Lemma 3.2: The portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods if and only if the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$.

Proof: The proof is in the Appendix B. ■

In the following lemma, we demonstrate that if the condition $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ is satisfied for each $z \in \mathcal{Z}^+$, then for any element $m \in \mathcal{M} \cap (\alpha_2, \alpha_1)$, there exists an N-period market scenario where the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods and $\{Z(n) = Z^{(n)}\}_{n=1}^{N}$ such that $m = \sum_{n=1}^{N} Z^{(n)}$ for some $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ and $N \in \mathbb{N}$. It follows that the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ when the portfolio never leaves the interval $(b-\epsilon, b+\epsilon)$ for N investment periods is equivalent to the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. Hence, we show that the cardinality of the set of achievable portfolios is equal to the cardinality of the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. After this lemma, we present conditions under which the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is finite so that the set of achievable portfolios is also finite.

Lemma 3.3: If $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$, then any element of $\mathcal{M} \cap (\alpha_2, \alpha_1)$ can be written as a sum $\sum_{n=1}^{N} Z^{(n)}$ for some $N \in \mathbb{N}$ where $\{Z(n) = Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ and $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for

$k = 1, \ldots, N$.

Proof: In Lemma 3.1, we showed that for any investment period N, the number of different portfolio values that $b(N)$ can take is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take where $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. Since this is true for any investment period N, it follows that the number of all achievable portfolios is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} Z(n) \in (\alpha_2, \alpha_1)$.

Here, we show that if $m \in \mathcal{M} \cap (\alpha_2, \alpha_1)$, then there exists a sequence $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ for some $N \in \mathbb{N}$ such that $m = \sum_{n=1}^{N} Z^{(n)}$ and $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. Let $m \in \mathcal{M} \cap (\alpha_2, \alpha_1)$. Then, it can be written as $m = m_1 z_1 + \ldots + m_{M^+} z_{M^+}$ for some $m_i \in \mathbb{Z}$ and $z_i \in \mathcal{Z}^+$, $i = 1, \ldots, M^+$. We define $S(k) = \sum_{n=1}^{k} Z^{(n)}$ for $k \geq 1$ and construct a sequence $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ for some $N \in \mathbb{N}$ such that $m = \sum_{n=1}^{N} Z^{(n)}$ and $S(k) \in (\alpha_2, \alpha_1)$ for each $k = 1, \ldots, N$ as follows. We choose $z_i \in \mathcal{Z}^+$ such that $m_i > 0$, let $Z^{(1)} = z_i$ and decrease m_i by 1. We see that $S(1) = Z^{(1)} \in (\alpha_2, \alpha_1)$ since $z_i < \min\{|\alpha_1|, |\alpha_2|\}$. Next, we choose $z_j \in \mathcal{Z}^+$ such that $m_j < 0$, let $Z^{(2)} = -z_j$ and increase m_j by 1. Then, it follows that $S(2) = Z^{(1)} + Z^{(2)} = z_i - z_j \in (\alpha_2, \alpha_1)$ since $z_i, z_j < \min\{|\alpha_1|, |\alpha_2|\}$. At any time $k \geq 3$, if

- $S(k) \geq 0$, we choose $z_l \in \mathcal{Z}^+$ such that $m_l < 0$, let $Z^{(k+1)} = -z_l$ and increase m_l by 1. Note that $S(k+1) \in (\alpha_2, \alpha_1)$ since $S(k) \in (\alpha_2, \alpha_1)$, $S(k) \geq 0$ and $Z^{(k+1)} < 0$. Now assume that there exists no $z_l \in \mathcal{Z}^+$ such that $m_l < 0$, i.e., $m_j \geq 0$ for $j = 1, \ldots, M$. If we let $I \triangleq \{j \in \{1, \ldots, M\} \mid m_j \geq 0\} = \{k_1, \ldots, k_T\}$ where $T \triangleq |I|$ and
$$Z^{(l)} = z_{k_j}, \quad l = k + 1 + \sum_{i=1}^{j-1} k_i, \ldots, k + \sum_{i=1}^{j} k_i$$
for $j = 1, \ldots, T$, then we get that $m = S(N) = \sum_{n=1}^{N} Z^{(n)}$ where $N = k + \sum_{i=1}^{T} k_i$. We observe that $S_i \in (\alpha_2, \alpha_1)$ for $i = k+1, \ldots, N$ since $m \in (\alpha_2, \alpha_1)$, $\sum_{j=1}^{T} m_{k_j} x_{k_j} \geq 0$ and $S(k) > 0$.
- $S(k) < 0$, we choose $z_l \in \mathcal{Z}^+$ such that $m_l > 0$, let $Z^{(k+1)} = z_l$ and decrease m_l by 1. Note that $S(k+1) \in (\alpha_2, \alpha_1)$ since $S(k) \in (\alpha_2, \alpha_1)$, $S(k) < 0$ and $Z^{(k+1)} \geq 0$. Assume that there exists no $z_l \in \mathcal{Z}^+$ such that $m_l \geq 0$, i.e., $m_j < 0$ for $j = 1, \ldots, M$. If we let $J \triangleq \{j \in \{1, \ldots, M\} \mid m_j \leq 0\} = \{k_1, \ldots, k_W\}$ where $W \triangleq |J|$ and
$$Z^{(l)} = z_{k_j}, \quad l = k + 1 + \sum_{i=1}^{j-1} k_i, \ldots, k + \sum_{i=1}^{j} k_i$$
for $j = 1, \ldots, W$, then we get that $m = S(N) = \sum_{n=1}^{N} Z^{(n)}$ where $N = k + \sum_{i=1}^{W} k_i$. We see that $S_i \in (\alpha_2, \alpha_1)$ for $i = k+1, \ldots, N$ since $m \in (\alpha_2, \alpha_1)$, $\sum_{j=1}^{W} m_{k_j} x_{k_j} \leq 0$ and $S(k) < 0$.

Therefore, we can write $m = \sum_{n=1}^{N} Z^{(n)}$ for some $N \geq 1$ where $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ and $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. ∎

Hence, we showed that if the condition $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ is satisfied for each $z \in \mathcal{Z}^+$, then any element of the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ can be written as a sum $\sum_{n=1}^{N} Z(n)$ for some $N \in \mathbb{N}$ when the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$ for N investment periods. It follows that the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ when the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$ for N investment periods is equivalent to the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. Thus, the number of achievable portfolios is equal to the cardinality of the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. In the following theorem, we demonstrate that if $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$ and the set $\mathcal{M}$ has a minimum positive element, then $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is finite. Hence, the set of achievable portfolios is also finite under these conditions. Otherwise, we show that the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ contains infinitely many elements so that the set of achievable portfolios is also infinite. Thus, we show that the set of achievable portfolios is finite if and only if the minimum positive element of the set $\mathcal{M}$ exists.

Theorem 3.1: If $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$ and the set $\mathcal{M}$ has a minimum positive element, i.e., if

$$\delta = \min\{m \in \mathcal{M} \mid m > 0\}$$

exists, then the set of achievable portfolio $\mathcal{B} = \cup_{n=1}^{\infty} \mathcal{B}_n$ is finite. If such a minimum positive element does not exist, then $\mathcal{B}$ is countably infinite.

In Theorem 3.1 we present a necessary and sufficient condition for the achievable portfolios to be finite. We emphasize that the required condition, i.e., $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$, is a necessary required technical condition which assures that the TRP thresholds are large enough to prohibit constant rebalancings at each investment period. In this sense, this condition does not limit the generality of the TRP framework.

By Theorem 3.1, we establish the conditions for a unique stationary distribution of the achievable portfolios. With the existence of a unique stationary distribution, in the next section, we provide the asymptotic behavior of the expected wealth growth by presenting the growth rate.

Proof: For any investment period N, we showed in Lemma 3.1 that the number of different portfolio values that $b(N)$ can take is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take where the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. In the Lemma 3.3, we showed that the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take where the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$ is equivalent to the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. We let $\mathcal{H}$ be the set of values that the sum $\sum_{n=1}^{N} Z(n) \in (\alpha_2, \alpha_1)$ can take for any $N \in \mathbb{N}$, i.e., $\mathcal{H} = \{\sum_{n=1}^{N} Z^{(n)} \mid \{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z},\ \sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1) \text{ for } k = 1, \ldots, N,\ N \in \mathbb{N}\}$. Now, assume that the minimum positive element δ exists. We next illustrate that the sum $\sum_{n=1}^{N} Z^{(n)}$ for any sequence $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ can be written as $k\delta$ for some $k \in \mathbb{Z}$, i.e., $\sum_{n=1}^{N} Z^{(n)} = k\delta$.

Assume that there exists a sequence $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ such that the sum $Z = \sum_{n=1}^{N} Z^{(n)} \neq k\delta$ for any $k \in \mathbb{Z}$. If we divide the real line into intervals of length δ, then Z should lie in one of the intervals, i.e., there exists $k_0 \in \mathbb{Z}$ such that $k_0\delta < Z < (k_0+1)\delta$ so that we can write $Z = k_0\delta + \eta$ where $0 < \eta < \delta$. By definition of $\mathcal{M}$, an integer multiple of any element of $\mathcal{M}$ is also an element of $\mathcal{M}$ so that $k_0\delta \in \mathcal{M}$ since $\delta \in \mathcal{M}$. Moreover, for any two elements of $\mathcal{M}$, their difference is also an element of $\mathcal{M}$ so that $\eta = Z - k_0\delta \in \mathcal{M}$ since $Z \in \mathcal{M}$ and $k_0\delta \in \mathcal{M}$. However, this contradicts to the fact that δ is the minimum positive element of $\mathcal{M}$ since $0 < \eta < \delta$ and $\eta \in \mathcal{M}$. Hence, it follows that any element of $\mathcal{H}$ can be written as $k\delta$ for some $k \in \mathbb{Z}$. Note that there are finitely many elements in $\mathcal{H}$ since any element $h \in \mathcal{H}$ can be written as $h = k\delta$ for some $k \in \mathbb{Z}$ and $\alpha_2 < h < \alpha_1$. Since $|\mathcal{B}| = |\mathcal{H}|$, it follows that the set of achievable portfolios $\mathcal{B}$ is finite.

To show that if δ does not exist then $\mathcal{B}$ contains infinitely many elements, we assume that δ does not exist. Since every finite set of real numbers has a minimum, there are either countably infinitely many positive elements in the set $\mathcal{M}$ or none. We know that there exists $z_i \neq 0$ so that there are positive numbers in $\mathcal{M}$. Therefore, there are infinitely many elements in $\mathcal{M}$. Now assume that there exists $\gamma_1 > 0$ that can be written as a sum $\sum_{n=1}^{N} Z^{(n)}$ for some $N \in \mathbb{N}$ where $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ and $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$. Then, by Lemma 3.3, it follows that $\gamma_1 \in \mathcal{M} \cap (0, \alpha_1)$ and since there exists no positive minimum element of $\mathcal{M}$, there exists $\gamma_2 > 0$ such that $\gamma_2 < \gamma_1$ so that $\gamma_2 \in \mathcal{M} \cap (0, \alpha_1)$. In this way, we can construct a decreasing sequence $\{\gamma_n\}$ such that $\gamma_n \in \mathcal{M} \cap (0, \alpha_1)$ for each $n \in \mathbb{N}$. Note that for any $n \in \mathbb{N}$, γ_n is also element of $\mathcal{H}$ by Lemma 3.3 so that there are countably infinite elements in $\mathcal{H}$. Hence, it follows that $\mathcal{B}$ has countably infinitely many elements. ■

We showed that if $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$ and the minimum positive element of the set $\mathcal{M}$ exists, then the set of achievable portfolios, $\mathcal{B}$, is finite. If the minimum positive element of the set $\mathcal{M}$ does not exist, then the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is countably infinite so that the number of achievable portfolios is also countably infinite. Hence, the set of achievable portfolios is finite if and only if the minimum positive element of the set $\mathcal{M}$ exists. However, Theorem 3.1 does not specify the exact number of achievable portfolios. In the following corollary, we demonstrate that the number of achievable portfolios is $\lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$ if the set of achievable portfolios is finite.

Corollary 3.1: If $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$ and $\delta = \min\{m | m > 0\, m \in \mathcal{M}\}$ exists, then the number of achievable portfolios is $\lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$[1].

Proof: Assume that δ exists and there exists $\theta > 0$ such that θ can be written as a sum $\sum_{n=1}^{N} Z^{(n)}$

[1]Here, $\lfloor x/y \rfloor$ is the largest integer less than or equal to x/y

for some $N \in \mathbb{N}$ and $\{Z(n) = Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ such that $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. Note that such a θ exists, e.g., $\theta = z > 0$ where $z \in \mathcal{Z}^+$ since $z \in (\alpha_2, \alpha_1)$. Then, by Lemma 3.3, it follows that $\theta \in \mathcal{M} \cap (0, \alpha_1)$. Since δ is the minimum positive element of $\mathcal{M}$, it follows that $0 < \delta \leq \theta$ and $\delta \in \mathcal{M} \cap (0, \alpha_1)$. Hence, by Lemma 3.3, we get that δ can be written as a sum $\sum_{n=1}^{N'} Z^{(n)}$ for some $N' \in \mathbb{N}$ and $\{Z^{(n)}\}_{n=1}^{N'} \in \mathcal{Z}$ where $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N'$. We note that δ is an element of the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ and $Z(n) \in \mathcal{Z}$ for $n = 1, \ldots, N$ such that the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$. We showed in Theorem 3.1 that any element of $\mathcal{M}$ can be written as $k\delta$ for some $k \in \mathbb{Z}$ so that the number of elements in $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is $\lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$. Hence, it follows that there are exactly $\lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$ achievable portfolios since Lemma 3.3 implies that the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is equivalent to the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ and $Z(n) \in \mathcal{Z}$ for $n = 1, \ldots, N$ such that the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for each $k = 1, \ldots, N$ and the cardinality of the latter set is equal to the number of achievable portfolios. ■

In Theorem 3.1, we introduce conditions on the cardinality of the set of all achievable portfolio states, $\mathcal{B}$, and showed that if $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for all $z \in \mathcal{Z}^+$ and the minimum positive element of the set $\mathcal{M}$ exists, then $\mathcal{B}$ is finite. This result is significant when we analyze the asymptotic behavior of the expected achieved wealth, i.e., in the following, we demonstrate that when $\mathcal{B}$ is finite, the portfolio sequence converges to a stationary distribution. Hence, we can determine the limiting behavior of the expected achieved wealth so that we can optimize b and ϵ. To accomplish this, specifically, we first present a recursive update to evaluate $E[S(n)]$. We then maximize $g(b, \epsilon) \overset{\triangle}{=} \lim_{n \to \infty} \frac{1}{n} \log E[S(n)]$ over b and ϵ with a brute-force search, i.e., we calculate $g(b, \epsilon)$ for different (b, ϵ) pairs and find the one that yields the maximum.

C. Finite State Markov Chain for Threshold Portfolios

If we assume that $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for all $z \in \mathcal{Z}^+$ and $\delta = \min\{m \in \mathcal{M} \mid m > 0\}$ exists, then the set of all achievable portfolios $\mathcal{B}$ is finite. By Corollary 3.1, it follows that there are exactly $L = \lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$ achievable portfolios. We let $\mathcal{B} = \{b_1, \ldots, b_L\}$ and, without loss of generality, $b_1 = b$. We define the probability mass vector of the portfolio sequence as $\boldsymbol{\pi}(n) = [\pi_1(n) \ \ldots \ \pi_L(n)]^T$ where $\pi_i(n) \overset{\triangle}{=} \mathrm{P}\left(b(n) = b_i\right)$. The portfolio sequence $b(n)$ forms a homogeneous Markov chain with a finite state space $\mathcal{B}$ since the transition probabilities between states are independent of period n. We see that $b(n)$ is irreducible since each state communicates with other states so that all states are null-persistent since $\mathcal{B}$ is finite [32]. Then, it follows that there exists a unique stationary distribution vector $\boldsymbol{\pi}$, i.e., $\boldsymbol{\pi} = \lim_{n \to \infty} \boldsymbol{\pi}(n)$. To calculate $\boldsymbol{\pi}$, we first observe that the set of portfolios that are connected to b_l, $\mathcal{N}_{l,n}$, and the set of price relative vectors that connect b_k to b_l, $\mathcal{U}_{k,l,n}$, are independent of investment period since the price relative sequences are i.i.d. for $k = 1, \ldots, L$ and $l = 1, \ldots, L$. Hence, we write $\mathcal{U}_{k,l,n} = \mathcal{U}_{k,l}$ and $\mathcal{N}_{l,n} = \mathcal{N}_l$ for $n \in \mathbb{N}$. We next note that the state transition probabilities are also independent of investment period and write $q_{k,l,n} = \mathrm{P}\left(b(n) = b_l | b(n-1) = b_k\right) = q_{k,l}$ for $n \in \mathbb{N}$, $k = 1, \ldots, L$ and $l = 1, \ldots, L$. Therefore, we can write $\mathrm{P}\left(b(n) = b_l\right)$ as

$$\mathrm{P}\left(b(n) = b_l\right) = \sum_{b_k \in \mathcal{N}_l} q_{k,l} \mathrm{P}\left(b(n-1) = b_k\right) = \sum_{k=1}^{L} q_{k,l} \mathrm{P}\left(b(n-1) = b_k\right), \tag{14}$$

where $q_{k,l} = 0$ if $b_k \notin \mathcal{N}_l$. Now, by using the definition of $\boldsymbol{\pi}(n)$ and (14), we get $\boldsymbol{\pi}(n+1) = \mathbf{P}\boldsymbol{\pi}(n)$ for each n, where $\mathbf{P}$ is the state transition matrix, i.e., $\mathbf{P}_{ij} = q_{i,j}$.

We next determine the limiting behavior of the expected achieved wealth $E[S(n)]$ to optimize b and ϵ as follows. In Section III-A, we showed that $E[S(n)]$ can be calculated iteratively by (2), (7) and (12). If we define the vector $\mathbf{e}(n) = [e_1(n) \ \ldots \ e_L(n)]^T$ where $e_i(n) \triangleq \mathrm{P}\left(b(n) = b_i\right) E[S(n) | b(n) = b_i]$, then we can calculate $E[S(n)]$ as the sum of the entries of $\mathbf{e}(n)$ by (2), i.e.,

$$E[S(n)] = \sum_{i=1}^{L} \mathrm{P}\left(b(n) = b_i\right) E[S(n) | b(n) = b_i] = \sum_{i=1}^{L} e_i(n) = \mathbf{1}^T \mathbf{e}(n), \tag{15}$$

where $\mathbf{1}$ is the vector of ones. Hence, by definition of $\mathbf{e}(n)$, we can write

$$\mathbf{e}(n+1) = \mathbf{Q}\mathbf{e}(n), \tag{16}$$

where the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \tag{17}$$

$$\begin{bmatrix} \sum\limits_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{1,1}} \left(b_1 w_1 + (1-b_1) w_2\right) p_1(w_1) p_2(w_2) & \cdots & \sum\limits_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{L,1}} \left(b_L w_1 + (1-b_L) w_2\right) p_1(w_1) p_2(w_2) \\ \vdots & \ddots & \vdots \\ \sum\limits_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{1,L}} \left(b_1 w_1 + (1-b_1) w_2\right) p_1(w_1) p_2(w_2) & \cdots & \sum\limits_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{L,L}} \left(b_L w_1 + (1-b_L) w_2\right) p_1(w_1) p_2(w_2) \end{bmatrix}$$

where we ignore rebalancing for presentation purposes. From (7) and (12), $\mathbf{Q}$ does not depend on period n since there are finitely many portfolio states, i.e., $\mathbf{Q}$ is constant. If we take rebalancing into account, then only the first row of the matrix $\mathbf{Q}$ changes and the other rows remain the same where

$$\begin{aligned} \mathbf{Q}_{1,j} = & \sum_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{V}_{j,1}} \left(b_1 w_1 + (1-b_1) w_2\right) p_1(w_1) p_2(w_2) \\ & + \sum_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{R}_{j,1}} \left(b_1 w_1 + (1-b_1) w_2\right) \left(1 - c \left| \frac{b_1 w_1}{b_1 w_1 + (1-b_1) w_2} - b \right| \right) p_1(w_1) p_2(w_2), \end{aligned}$$

$\mathcal{V}_{j,1}$ is the set of price relative vectors that connect b_j to $b_1 = b$ without crossing the threshold boundaries and $\mathcal{R}_{j,1}$ is the set of price relative vectors that connect b_j to $b_1 = b$ by crossing the threshold boundaries for $i = j, \ldots, L$. Note that we can find the matrix $\mathbf{Q}$ by using the set of achievable portfolios $\mathcal{B}$ and the probability mass vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of the price relative sequences.

Here, we analyze $E[S(n)]$ as $n \to \infty$ as follows. We assume that the matrix $\mathbf{Q}$ is diagonalizable with the eigenvalues $\lambda_1, \ldots, \lambda_L$ and, without loss of generality, $\lambda_1 \geq \ldots \geq \lambda_L$, which is the case for a wide range of transaction costs [33]. Then, there exists a nonsingular matrix $\mathbf{B}$ such that $\mathbf{Q} = \mathbf{B}\mathbf{\Lambda}\mathbf{B}^{-1}$ where $\mathbf{\Lambda}$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_L$. We observe that the matrix $\mathbf{Q}$ has nonnegative entries. Therefore, it follows from Perron-Frobenius Theorem [34] that the matrix $\mathbf{Q}$ has a unique largest eigenvalue $\lambda_1 > 0$ and any other eigenvalue is strictly smaller than λ_1 in absolute value, i.e., $\lambda_1 > |\lambda_j|$ for $j = 2, \ldots, L$. Then, the recursion (16) yields

$$\mathbf{e}(n) = \mathbf{Q}^n \mathbf{e}(0) = \mathbf{B}\mathbf{\Lambda}^n\mathbf{B}^{-1}\mathbf{e}(0) = \mathbf{B} \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_L^n \end{bmatrix} \mathbf{B}^{-1}\mathbf{e}(0).$$

Hence, the expected achieved wealth $E[S(n)]$ is given by

$$E[S(n)] = \mathbf{1}^T\mathbf{e}(n) = \mathbf{1}^T\mathbf{B} \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_L^n \end{bmatrix} \mathbf{B}^{-1}\mathbf{e}(0) = \mathbf{u}^T \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_L^n \end{bmatrix} \mathbf{v}$$

$$= \sum_{i=1}^{L} u_i v_i \lambda_i^n,$$

where $\mathbf{u} \triangleq [u_1 \ \ldots \ u_L]^T = \mathbf{B}^T\mathbf{1}$ and $\mathbf{v} \triangleq [v_1 \ \ldots \ v_L] = \mathbf{B}^{-1}\mathbf{e}(0)$. Then, it follows that

$$g(b,\epsilon) = \lim_{n\to\infty} \frac{1}{n} \log E[S(n)] = \lim_{n\to\infty} \frac{1}{n} \log \left\{ \sum_{i=1}^{L} u_i v_i \lambda_i^n \right\} = \lim_{n\to\infty} \frac{1}{n} \log \left\{ \lambda_1^n \left[\sum_{i=1}^{L} u_i v_i \left(\frac{\lambda_i}{\lambda_1} \right)^n \right] \right\}$$

$$= \lim_{n\to\infty} \log \lambda_1 + \lim_{n\to\infty} \frac{1}{n} \log \left\{ \sum_{i=1}^{L} u_i v_i \left(\frac{\lambda_i}{\lambda_1} \right)^n \right\}$$

$$= \log \lambda_1$$

since $\lim_{n\to\infty} \left(\frac{\lambda_i}{\lambda_1}\right)^n = 0$ for $i = 2, \ldots, L$. Hence, we can optimize b and ϵ as

$$[b^*, \epsilon^*] = \arg\max_{b\in[0,1],0<\epsilon} g(b,\epsilon) = \arg\max_{b\in[0,1],0<\epsilon} \log \lambda_1.$$

To maximize $g(b,\epsilon)$, we evaluate it for different values of (b,ϵ) pairs and find the pair that maximizes $g(b,\epsilon)$, i.e., by a brute-force search in the Simulations section.

In this section, we first demonstrated that the set of achievable portfolios is finite under certain conditions. We then showed that the portfolio forms a Markov chain with a finite state space and find the corresponding transition matrix and the stationary state probabilities. When $\mathcal{B}$ is finite, we derived a recursive update with a constant complexity, i.e., the number of states does not grow, to calculate the expected achieved wealth. Finally, we investigated the asymptotic behavior of the expected achieved wealth using this update to optimize b and ϵ with a brute-force search.

In the next section, we investigate the well-studied two-asset Brownian market model with transaction costs. We first show that the set of achievable portfolios is finite and calculate the state transition probabilities. Then, we calculate the asymptotic behavior of the expected achieved wealth to optimize b and ϵ.

D. Two Stock Brownian Markets

In this section, we consider the well-known two-asset Brownian market, where stock price signals are generated from a standard Brownian motion [19], [20], [22]. Portfolio selection problem in continuous time two-asset Brownian markets with proportional transaction costs was investigated in [20], where the growth optimal investment strategy is shown to be a threshold portfolio. Here, as usually done in the financial literature [19], we first convert the continuous time Brownian market by sampling to a discrete time market [22]. Then, we calculate the expected achieved wealth and optimize b and ϵ to find the best portfolio rebalancing strategy for a discrete-time Brownian market with transaction costs. Note that although, the growth optimal investment in discrete-time two-asset Brownian markets with proportional transaction costs was investigated in [22], the expected achieved wealth and the optimal threshold interval $(b-\epsilon, b+\epsilon)$ has not been calculated yet.

To model the Brownian two-asset market, we use the price relative vector $\mathbf{X} = [X_1 \; X_2]^T$ with $X_1 = 1$ and $X_2 = e^{kZ}$ where k is constant and Z is a random variable with $\mathrm{P}\left(Z = \pm 1\right) = \frac{1}{2}$. This price relative vector is obtained by sampling the stock price processes of the continuous time two-asset Brownian market [20], [22]. We emphasize that this sampling results a discrete-time market identical to the binomial model popular in asset pricing [22]. We first present the set of achievable portfolios and the transition probabilities between portfolio states. We then investigate the asymptotic behavior of the expected achieved wealth to optimize b and ϵ.

Since the price of the first stock is the same over investment periods, the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ if either the money in the second stock grows over a certain limit or falls below a certain limit. If the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods, then the money in the first stock is b dollars and the money in the second stock is $(1-b)e^{ki}$ for some $-N \leq i \leq N$ so that the portfolio is $b(N) = \frac{b}{b+(1-b)e^{ki}}$. Note that $\frac{b}{b+(1-b)e^{ki}} \in (b-\epsilon, b+\epsilon)$ if and only if $i_{\min} \leq i \leq i_{\max}$, where $i_{\min} \triangleq \left\lceil \frac{1}{k} \ln \frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} \right\rceil$[2] and $i_{\max} \triangleq \left\lfloor \frac{1}{k} \ln \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)} \right\rfloor$. Hence, the set of achievable portfolios is given by

$$\mathcal{S} = \left\{ b_i = \frac{b}{b + (1-b)e^{(i+i_{\min}-1)k}} \mid i = 1, \ldots, i_{\max} - i_{\min} + 1 \right\} = \{b_1, \ldots, b_S\},$$

where $|\mathcal{S}| = S$ and $S \triangleq i_{\max} - i_{\min} + 1$ and $b_{1-i_{\min}} = b$. We see that the portfolio is rebalanced to $b_{1-i_{\min}} = b$ only if it is in the state b_1 and $X_2 = e^{-k}$ or if it is in the state b_S and $X_2 = e^{k}$. Therefore, the transition probabilities are given by

$$\mathrm{P}\left(b_i | b_j\right) = \begin{cases} \frac{1}{2} & : \ i = 2, \ldots, S-1 \text{ and } j = i \pm 1 \ \text{, or } \ i = 1 \text{ and } j \in \{2, 1 - i_{\min}\}, \ \text{ or } \ i = S \text{ and } j \in \{S-1, 1 - i_{\min}\} \\ 0 & : \ \text{otherwise,} \end{cases}$$

where $P(b_i|b_j)$ is the probability that the portfolio $b(n) = b_i$ given that $b(n-1) = b_j$ for any period n. We now calculate $E[S(n)]$ using (15) and (16) as follows. The sets of price relative vectors that connect portfolio states are given by

$$\mathcal{U}_{i,j} = \begin{cases} \{\left[1 \ e^{k}\right]^T\} & : \ i = 1, \ldots, S-1 \text{ and } j = i+1, \ \text{ or } \ i = S \text{ and } j = 1 - i_{\min} \\ \{\left[1 \ e^{-k}\right]^T\} & : \ i = 2, \ldots, S-1 \text{ and } j = i-1, \ \text{ or } \ i = 1 \text{ and } j = 1 - i_{\min} \\ \varnothing & : \ \text{otherwise.} \end{cases}$$

Hence, we can calculate the matrix $\mathbf{Q}$ defined in (17) as

$$\mathbf{Q}_{i,j} = \begin{cases} \frac{1}{2}(b_j + (1-b_j)e^{k}) & : \ i = 2, \ldots, S \text{ and } j = i-1 \\ \frac{1}{2}(b_j + (1-b_j)e^{-k}) & : \ i = 1, \ldots, S-1 \text{ and } j = i+1 \\ 0 & : \ \text{otherwise,} \end{cases}$$

where we ignore rebalancing. If we take rebalancing into account, then

$$\mathbf{Q}_{1-i_{\min},1} = \frac{1}{2}(b_1 + (1-b_1)e^{-k}) \left(1 - c\left|\frac{b_1}{b_1 + (1-b_1)e^{-k}} - b\right|\right)$$

and

$$\mathbf{Q}_{1-i_{\min},S} = \frac{1}{2}(b_S + (1-b_S)e^{k}) \left(1 - c\left|\frac{b_S}{b_S + (1-b_S)e^{k}} - b\right|\right).$$

Then, by (15) and (16), $E[S_n]$ is given by $\mathbf{Q}^n \mathbf{e}(0)$. Moreover, we maximize

$$g(b, \epsilon) = \lim_{n \to \infty} \frac{1}{n} \log E[S_n] = \log \lambda_1,$$

where λ_1 is the largest eigenvalue of the matrix $\mathbf{Q}$. Here, we optimize b and ϵ with a brute-force search, i.e., we find λ_1 for different (b, ϵ) pairs and find the one that achieves the maximum.

[2]Here, $\lceil x/y \rceil$ is the largest integer greater or equal to the x/y.

In the next section, we sequentially estimate the probability mass vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of the price relative sequences $X_1(n)$ and $X_2(n)$ using a maximum likelihood estimator.

IV. Maximum Likelihood Estimators of The Probability Mass Vectors

In this section, we sequentially estimate the probability mass vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ corresponding to $X_1(n)$ and $X_2(n)$, respectively, using a maximum likelihood estimator (MLE). In general, these vectors may not be known or change in time, hence, could be estimated at each investment period prior to calculation of $E[S(n)]$. The maximum likelihood estimator for a pmf on a finite set is well-known [33], but we provide the corresponding derivations here for completeness. We consider, without loss of generality, the price relative sequence $X_1(n)$ and assume that its realizations are given by $X_1(n) = w_n \in \mathcal{X}$ for $n = 1, \ldots, N$ and estimate $\mathbf{p}_1$. Similar derivations follow for the price relative sequence $X_2(n)$ and $\mathbf{p}_2$. Note that as demonstrated in the Simulations section, the corresponding estimation can be carried out over a finite length window to emphasize the most recent data. We define the realization vector $\mathbf{w} = [w_1, \ldots, w_N]$ and the probability mass function as $p_\theta(x_i) = p_1(x_i|\theta) = \theta_{x_i}$ for $i = 1, \ldots, K$ and the parameter vector $\theta \triangleq [\theta_{x_1}, \ldots, \theta_{x_K}]$. Then, the MLE of the probability mass vector $\mathbf{p}_1$ is given by

$$\theta_{\mathrm{MLE}} = \underset{\theta:\sum_{i=1}^{K}\theta_{x_i}=1}{\arg\max}\; p_1(\mathbf{w}|\theta) = \underset{\theta:\sum_{i=1}^{K}\theta_{x_i}=1}{\arg\max}\; \mathrm{P}\left(X_1(1) = w_1, \ldots, X_1(N) = w_N|\theta\right). \tag{18}$$

Since the price relative sequence $X_1(n)$ is i.i.d., it follows that

$$p_1(\mathbf{w}|\theta) = \prod_{i=1}^{N} p_1(w_i|\theta) = \prod_{i=1}^{N}\theta_{w_i} = \prod_{i=1}^{N}\prod_{j=1}^{K}\theta_{x_j}^{\mathrm{I}(w_i=x_j)}, \tag{19}$$

where (19) follows since $\mathrm{I}(.)$ is the indicator function, i.e., $\mathrm{I}(w_i = x_j) = 1$ if $w_i = x_j$ and $\mathrm{I}(w_i = x_j) = 0$ if $w_i \neq x_j$. If we change the order of the product operators in (19), then we obtain

$$p_1(\mathbf{w}|\theta) = \prod_{i=1}^{N}\prod_{j=1}^{K}\theta_{x_j}^{\mathrm{I}(w_i=x_j)} = \prod_{j=1}^{K}\prod_{i=1}^{N}\theta_{x_j}^{\mathrm{I}(w_i=x_j)} = \prod_{j=1}^{K}\theta_{x_j}^{\sum_{i=1}^{N}\mathrm{I}(w_i=x_j)} = \prod_{j=1}^{K}\theta_{x_j}^{N_j},$$

where $N_j \triangleq \sum_{i=1}^{N}\mathrm{I}(w_i = x_j)$, i.e., the number of realizations that are equal to $x_j \in \mathcal{X}$ for $j = 1, \ldots, K$. Note that $\sum_{j=1}^{K} N_j = N$. Hence, we can write (18) as

$$\begin{aligned}\theta_{\mathrm{MLE}} &= \underset{\theta:\sum_{i=1}^{K}\theta_{x_i}=1}{\arg\max}\; p_1(\mathbf{w}|\theta) = \underset{\theta:\sum_{i=1}^{K}\theta_{x_i}=1}{\arg\max}\prod_{j=1}^{K}\theta_{x_j}^{N_j} = \underset{\theta:\sum_{i=1}^{K}\theta_{x_i}=1}{\arg\max}\frac{1}{N}\log\left(\prod_{j=1}^{K}\theta_{x_j}^{N_j}\right) \\ &= \underset{\theta:\sum_{i=1}^{K}\theta_{x_i}=1}{\arg\max}\sum_{j=1}^{K}\frac{N_j}{N}\log\theta_{x_j},\end{aligned} \tag{20}$$

where (20) follows that $\log(.)$ is a monotone increasing function. If we define the vector $\mathbf{h} = [h_{x_1}, \ldots, h_{x_K}]$, where $h_{x_j} \triangleq \frac{N_j}{N}$ for $j = 1, \ldots, K$, then we see that $h_{x_j} \geq 0$ for $j = 1, \ldots, K$ and $\sum_{j=1}^{K} h_{x_j} = 1$.

Since $\mathbf{h}$ and θ are probability vectors, i.e., their entries are nonnegative and sum to one, it follows that $\mathrm{D}(\mathbf{h}\|\theta) \triangleq \sum_{i=1}^{K} h_{x_j} \log\left(\frac{h_{x_j}}{\theta_{x_j}}\right) \geq 0$ and $\mathrm{D}(\mathbf{h}\|\theta) = 0$ if and only if $\theta = \mathbf{h}$, i.e., their relative entropy is nonnegative [35]. Therefore, we get that

$$\sum_{j=1}^{K} \frac{N_j}{N} \log \theta_{x_j} = \sum_{j=1}^{K} h_{x_j} \log \theta_{x_j} = \sum_{j=1}^{K} h_{x_j} \log\left(\frac{\theta_{x_j}}{h_{x_j}}\right) + \sum_{j=1}^{K} h_{x_j} \log h_{x_j}$$
$$= -\mathrm{D}(\mathbf{h}\|\theta) + \sum_{j=1}^{K} h_{x_j} \log h_{x_j} \leq \sum_{j=1}^{K} h_{x_j} \log h_{x_j},$$

where the equality is reached if and only if $\theta = \mathbf{h}$. Hence, it follows that

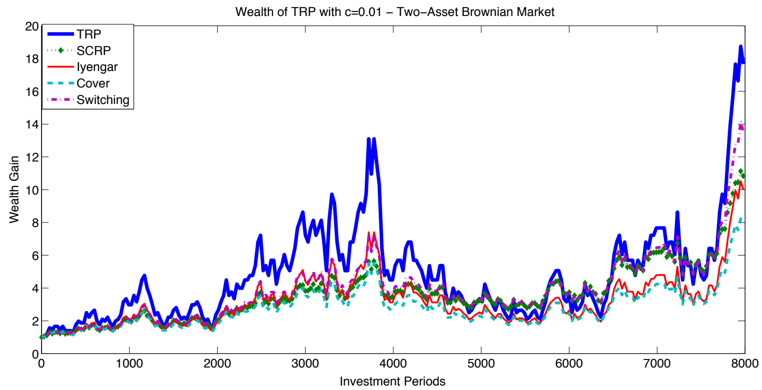
$$\theta_{\mathrm{MLE}} = \underset{\theta:\sum_{i=1}^{K}\theta_{x_i}=1}{\arg\max}\; p_1(\mathbf{w}|\theta) = \underset{\theta:\sum_{i=1}^{K}\theta_{x_i}=1}{\arg\max} \sum_{j=1}^{K} \frac{N_j}{N} \log \theta_{x_j} = \mathbf{h}$$

so that we estimate the probability mass vector $\mathbf{p}_1$ with $\mathbf{h} = \left[\frac{N_1}{N}, \ldots, \frac{N_K}{N}\right]$ at each investment period N where $\frac{N_j}{N}$ is the proportion of realizations up to period N that are equal to x_j for $x_j \in \mathcal{X}$.

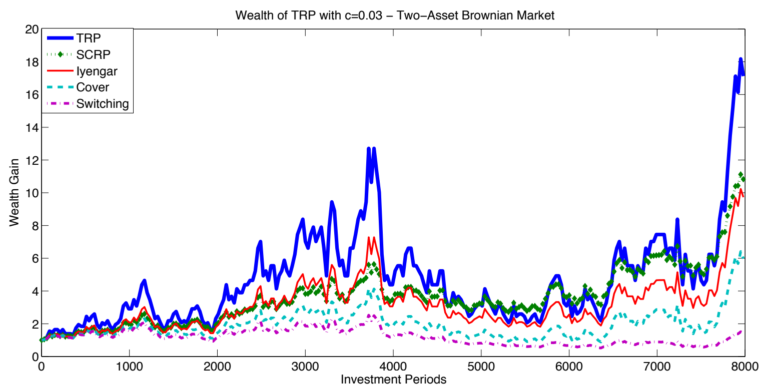
V. Simulations

In this section, we demonstrate the performance of TRPs with several different examples. We first analyze the performance of TRPs in a discrete-time two-asset Brownian market introduced in Section III-D. As the next example, we apply TRPs to historical data from [18], [36] collected from the New York Stock Exchange over a 22-year period and compare the results to those obtained from other investment strategies [18], [23], [24], [36]. Using the historical data set, we first simulate the performance of TRPs, the semiconstant rebalanced portfolio (SCRP) [18], the Iyengar's algorithm [23], the Cover's algorithm [36] and the switching portfolio from [24] on a randomly selected stock pair. Finally, we then present the average performance of TRPs on randomly selected pairs of stocks and show that the performance of the TRP algorithm is significantly better than the portfolio investment strategies from [18], [23], [24], [36] in historical data sets as expected from Section III.

As the first scenario, we apply TRPs to a discrete-time two-asset Brownian market. Under this well studied market in the financial literature [14], the price relative vector is given by $\boldsymbol{X} = [X_1\; X_2]^T$, where $X_1 = 1$, $X_2 = e^{kZ}$ and $Z = \pm 1$ with equal probabilities and we set $k = 0.03$ [22]. Here, the sample spaces of the price relative sequences X_1 and X_2 are $\mathcal{X}_1 = \{1\}$ and $\mathcal{X}_2 = \{0.97, 1.03\}$, respectively, and $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 = \{x_1, x_2, x_3\}$, where $x_1 = 1$, $x_2 = 0.97$, $x_3 = 1.03$. Hence, the probability mass vectors of the price relative sequences X_1 and X_2 are given by $\mathbf{p}_1 = [1\; 0\; 0]^T$ and $\mathbf{p}_2 = [0\; 0.5\; 0.5]^T$, respectively. Based on this data, we evaluate the growth rate for different (b, ϵ) pairs to find the best TRP that maximizes the growth rate using the approach introduced in Section III-D, i.e., we form the matrix $\mathbf{Q}$ and evaluate the corresponding maximum eigenvalues to find the pair that achieves



(a)



(b)

Fig. 2: Performance of portfolio investment strategies in the two-asset Brownian market. (a) Wealth gain with the cost ratio $c = 0.01$. (b) Wealth gain with the cost ratio $c = 0.03$.

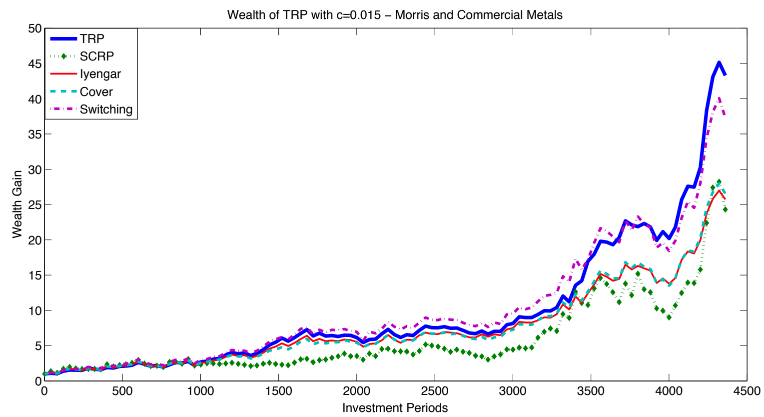
the largest maximum eigenvalue since this pair also maximizes the growth rate. Then, we invest 1 dollars in a randomly generated two-asset Brownian market using: the TRP, labeled as, "TRP", i.e., TRP(b,ϵ) with calculated (b, ϵ) pair, the SCRP algorithm with the target portfolio vector $\boldsymbol{b} = [0.5\ 0.5]$, labeled as "SCRP", as suggested in [18], the Iyengar's algorithm, labeled as "Iyengar", the Cover's algorithm, labeled as "Cover", and the switching portfolio, labeled as "Switching", with parameters suggested in [24]. In Fig. 2, we plot the wealth achieved by each algorithm for transaction costs $c = 0.01$ and $c = 0.03$, where c is the proportion paid when rebalancing, i.e, $c = 0.03$ is a 3% commission. As expected from the derivations in Section III, we observe that, in both cases, the performance of the TRP algorithm is significantly better than the other algorithms under transaction costs.

We next present results that illustrate the performance of TRPs on historical data sets [36]. As for the first example, we present results on the stock pair Morris and Commercial Metals (randomly selected) from the historical data sets [18], [36] for a mild transaction cost $c = 0.015$ and a hefty transaction cost $c = 0.03$ to better illustrate the effect of transaction costs. The data includes the price relative sequences of the stock pair for 5651 investment periods (days). Since the brute force algorithm introduced in Section III-A requires the sample spaces of the price relative sequences, we proceed as follows. We first calculate the sample spaces and the probability mass vectors of the price relative sequences from the first 1000-day realizations of X_1 and X_2, where the sample spaces are simply constructed by quantizing the observed realizations into bins. We observed that the performance of the TRP is not effected by the number of bins provided that there are an adequate number of bins to approximate the continuous valued price relatives. Then, we optimize b and ϵ using the MLE introduced in Section IV and the brute force algorithm from Section III, and invest using this TRP for the next 1000 periods, i.e., from period 1001 to period 2000. We then update (b, ϵ) pair using the first 2000-day realizations of the price relative vectors and invest using the best TRP for the next 1000 periods. We repeat this process through all available data. Hence, we invest on the two stocks using TRP for 4651 periods where we update (b, ϵ) pair at each 1000 periods. In Fig. 3, we present the performances of the TRP algorithm, the SCRP algorithm, the Iyengar's algorithm [23], the Cover's algorithm and the switching portfolio algorithm [24]. We observe that although the performance of the algorithms other than the TRP degrade with increasing transaction cost, the performance of the TRP, using the MLE, is not significantly effected since it can avoid excessive rebalancings. In both cases, the TRP readily outperforms the other simulated algorithms for these simulations.

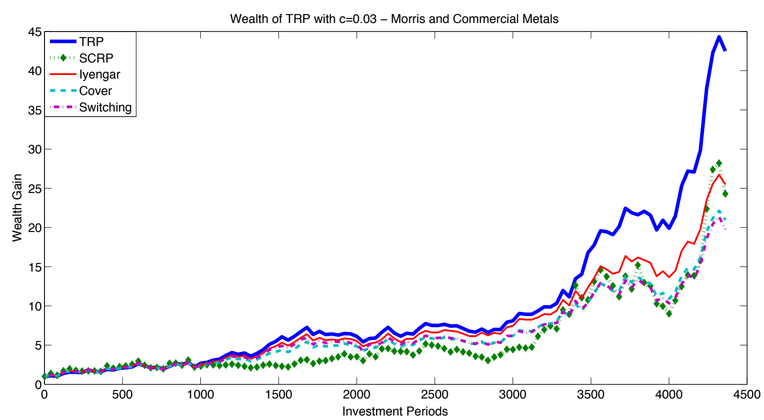
Finally, we illustrate the average performance of the threshold rebalancing strategy on a number of stock pairs to avoid any bias to particular stock pairs. In this set of simulations, we first randomly select pairs of stocks from the historical data that includes 34 stocks (where the Kin Ark stock is excluded) and invest using: the TRP algorithm, the SCRP algorithm, the Cover's algorithm, the Iyengar's algorithm and the switching portfolio, under a mild transaction cost $c = 0.015$ and a hefty transaction cost $c = 0.03$. In Fig. 4, we present the wealth gain for each algorithm, where the results are averaged over randomly selected 10 independent stock pairs. We observe from these simulations that the average performance of the TRP is better than the average performance of the other portfolio investment strategies commonly used in the literature.

VI. Conclusion

We studied growth optimal investment in i.i.d. discrete time two-asset markets under proportional transaction costs. Under this market model, we studied threshold portfolios that are shown to yield the optimal growth. We first introduced a recursive update to calculate the expected growth and then



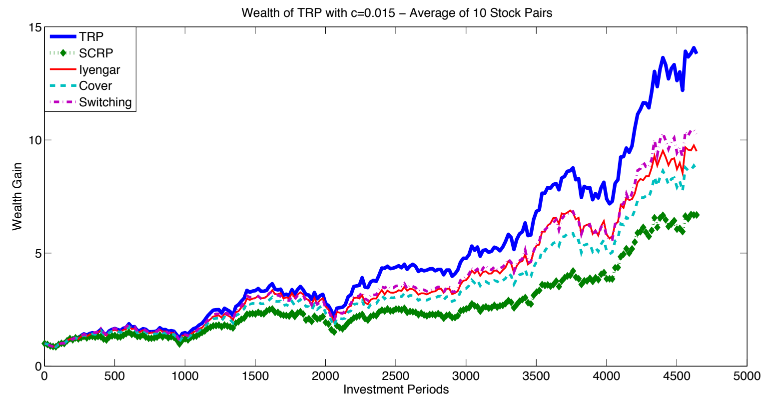
(a)



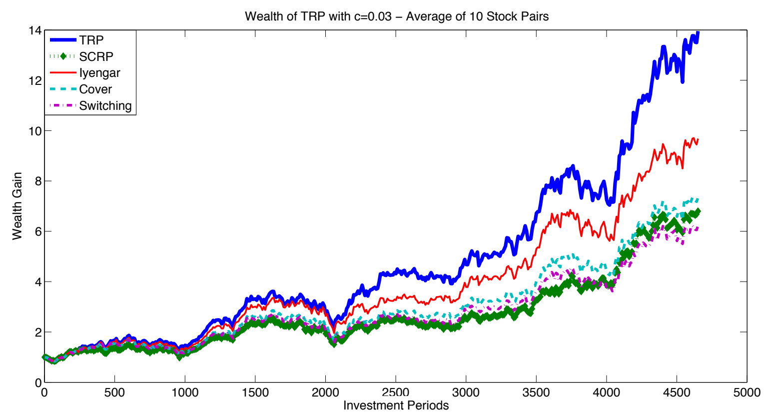
(b)

Fig. 3: Performance of portfolio investment strategies on the Morris-Commercial Metals stock pair. (a) Wealth gains with the cost ratio $c = 0.015$. (b) Wealth gains with the cost ratio $c = 0.03$.

demonstrated that under the threshold rebalancing framework, the achievable set of portfolios form an irreducible Markov chain under mild technical conditions. We evaluated the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized using a brute force approach yielding the growth optimal investment portfolio under proportional transaction costs in i.i.d. discrete-time two-asset markets. We also solved the optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets. For the case that the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator. We observed in our simulations, which include simulations using the historical data sets from [36], that the introduced



(a)



(b)

Fig. 4: Average performance of portfolio investment strategies on independent stock pairs. (a) Wealth gain with the cost ratio $c = 0.015$. (b) Wealth gain with the cost ratio $c = 0.03$.

TRP algorithm significantly improves the achieved wealth under both mild and hefty transaction costs as predicted from our derivations.

APPENDIX

A) Proof of Lemma 3.1:

We analyze the cardinality of the set $\mathcal{B}_N$ of achievable portfolios at period N, M_N, as follows. If we assume that an investor invests with a TRP(b,ϵ) for N investment periods and the sequence of price relative vectors are given by $\left\{[X_1(n)\ X_2(n)] = \left[X_1^{(n)}\ X_2^{(n)}\right]\right\}_{n=1}^{N}$ and the portfolio sequence is given by $\{b(n) = b_n\}_{n=1}^{N}$, then we see that the portfolio could leave the interval at any period depending on the realizations of the price relative vector. We define an N-period market scenario as a sequence of portfolios $\{b(n)\}_{n=1}^N$. We can find the number of achievable portfolios at period N as the number of different values that the last element of N-period market scenarios can take. Here, we partition the set of N-period market scenarios according to the last time the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ and show that any achievable portfolio at period N can be achieved by an N-period market scenario where the portfolio does no leave the interval $(b-\epsilon, b+\epsilon)$ for N periods as follows. If we define the set $\mathcal{P}$ as the set of N-period market scenarios, i.e.,

$$\mathcal{P} = \left\{\{b_n\}_{n=1}^N \mid b_n \in \mathcal{B}_n \ , n = 1, \ldots, N\right\} = \bigcup_{i=1}^{N+1} \mathcal{P}_i,$$

where $\mathcal{P}_i$ is the set of N-period market scenarios where the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ last time at period i, i.e.,

$$\mathcal{P}_i = \left\{\{b_n\}_{n=1}^N \mid b_n \in \mathcal{B}_n \ , n = 1, \ldots, N, \ b(n) \text{ leaves the interval } (b-\epsilon, b+\epsilon) \text{ last time at period } i\right\}$$

for $i = 1, \ldots, N$ and $\mathcal{P}_{N+1}$ is the set of N-period market scenarios where the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods, i.e.,

$$\mathcal{P}_{N+1} = \left\{\{b_n\}_{n=1}^N \mid b_n \in \mathcal{B}_n \ , n = 1, \ldots, N, \ b(n) \text{ never leaves the interval } (b-\epsilon, b+\epsilon) \text{ for } N \text{ periods}\right\}.$$

We point out that $\mathcal{P}_i$'s are disjoint, i.e., $\mathcal{P}_i \cap \mathcal{P}_j = \varnothing$ for $i \neq j$ and their union gives the set of all N-period market scenarios, i.e., $\bigcup_{i=1}^{N+1} \mathcal{P}_i = \mathcal{P}$ so that they form a partition for $\mathcal{P}$. We see that the set $\mathcal{B}_N$ of achievable portfolios at period N is the set of last elements of N-period market scenarios, i.e., $\mathcal{B}_N = \{b_N \mid \{b_n\}_{n=1}^N \in \mathcal{P}\}$. We next show that the last element of any N-period market scenario from $\mathcal{P}_i$ for $i = 1, \ldots, N$ is also a last element of an N-period market scenario from $\mathcal{P}_{N+1}$. Therefore, we demonstrate that any element of the set $\mathcal{B}_N$ is achievable by a market scenario from $\mathcal{P}_{N+1}$ and $\mathcal{B}_N = \{b_N \mid \{b_n\}_{n=1}^N \in \mathcal{P}_{N+1}\}$.

Assume that $\{b_n\}_{n=1}^N \in \mathcal{P}_i$ for some $i \in \{1, \ldots, N\}$ so that $b_i = b$, i.e., the portfolio is rebalanced to b last time at period i. Note that b_N can also be achieved by an N-period market scenario $\left\{b_n'\right\}_{n=1}^N$ where the portfolio never leaves the interval $(b-\epsilon, b+\epsilon)$, $b_j' = b_{i+j}$ for $j = 1, \ldots, N-i$ and $X_1^{(j)} = X_2^{(j)}$ for $j = N-i+1, \ldots, N$ so that $b_N' = b_{N-i}' = b_N$. Hence, it follows that the set of achievable portfolios at period N is the set of achievable portfolios by N-period market scenarios from $\mathcal{P}_{N+1}$. We next find the number of different values that $b(N)$ can take where the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods.

When the portfolio never leaves the interval $(b-\epsilon, b+\epsilon)$ for N investment periods, $b(N)$ is given by

$$b(N) = \frac{b \prod_{i=1}^N X_1(n)}{b \prod_{i=1}^N X_1(n) + (1-b) \prod_{i=1}^N X_2(n)}.$$

If we write the reciprocal of $b(N)$ as

$$\frac{1}{b(N)} = 1 + \frac{1-b}{b} \prod_{n=1}^{N} \frac{X_2(n)}{X_1(n)} = 1 + \frac{1-b}{b} e^{\sum_{n=1}^{N} Z(n)},$$

then we observe that the number of different values that the portfolio $b(N)$ can take is the same as the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take. Since the price relative sequences $X_1(n)$ and $X_2(n)$ are elements of the same sample space $\mathcal{X}$ with $|\mathcal{X}| = K$, it follows that $|\mathcal{Z}| = M \leq K^2 - K + 1$. Since the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take is equal to $\binom{N+M-1}{M-1}$ and $M \leq K^2 - K + 1$, it follows that the number of achievable portfolios at period N is bounded by $\binom{N+K^2-K}{K^2-K}$, i.e., $|\mathcal{B}_N| = M_N \leq \binom{N+K^2-K}{K^2-K}$ and the proof follows.

B) Proof of Lemma 3.2:

If the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods, then $b(n) \in (b-\epsilon, b+\epsilon)$ for $n = 1, \ldots, N$ and it is not adjusted to b at these periods so that

$$b(n) = \frac{b \prod_{i=1}^{n} X_1(i)}{b \prod_{i=1}^{n} X_1(i) + (1-b) \prod_{i=1}^{n} X_2(i)} \in (b-\epsilon, b+\epsilon)$$

for each $n = 1, \ldots, N$. Taking the reciprocal of $b(n)$, we get that

$$\frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} < \prod_{i=1}^{n} \frac{X_2(i)}{X_1(i)} < \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)}.$$

Noting that $\frac{X_2(i)}{X_1(i)} = e^{Z(i)}$ and taking the logarithm of each side, it follows that

$$\ln \frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} = \alpha_2 < \sum_{i=1}^{n} Z(i) < \ln \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)} = \alpha_1,$$

i.e., $\sum_{i=1}^{n} Z(i) \in (\alpha_2, \alpha_1)$ for $n = 1, \ldots, N$. Now, if the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ first time at period k for some $k \in \{1, \ldots, N\}$, then we get that $b(k) \geq b+\epsilon$ or $b(k) \leq b-\epsilon$ so that we get

$$\sum_{i=1}^{k} Z(i) \geq \alpha_1 \text{ or } \sum_{i=1}^{k} Z(i) \leq \alpha_2,$$

i.e., $\sum_{i=1}^{k} Z(i) \notin (\alpha_2, \alpha_1)$.

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