A CLASS OF LINEAR-PHASE REGULAR BIORTHOGONAL WAVELETS

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ABSTRACT

A class of biorthogonal systems leading to linear-phase wavelets is presented. A notable feature of this structure is that the wavelets are derived from a filter bank where the lowpass analysis filter is constrained to be a halfband filter. We derive FIR biorthogonal solutions from a *pair* of Lagrange halfband filters. We also consider IIR biorthogonal solutions based on a pair of zero-phase halfband filters derived from Butterworth halfband filters.

1. INTRODUCTION

Multiresolution signal representation based on wavelets or filter banks has drawn considerable attention recently [1]-[5] and found applications in areas such as image coding, edge detection, computer vision, texture discrimination, radar imaging, etc. Wavelets are functions whose translations and dilations provide an expansion of functions belonging to $L^2(\mathcal{R})$, and it is known that wavelet solutions have a close relationship with filter banks [2]. The perfect reconstruction (PR) filter bank, in general, provides biorthogonal solutions [6]-[8]. In a special case where the paraunitary condition is satisfied, the system is known to be orthonormal, and is related to the conjugate mirror filter bank [2, 9, 10]. When the phase linearity is required, the only possible paraunitary solution is the Haar solution.

In this paper we present a class of PR systems which leads to linear-phase wavelet solutions. In section 2 we consider a recently developed technique [11] for the synthesis of PR filter banks. Within this structure, in section 3 we then make a special selection of filter component based on a *pair* of Lagrange interpolation filters [12] which leads to linear-phase wavelets with the characteristics exhibited in regular wavelets. We also show that filter component based on Butterworth halfband filters [13] generates smooth wavelets.

2. FILTER BANKS

Consider the two-channel analysis/synthesis filter bank as shown in Figure 1. The z - transform of the output can be expressed as

$$\begin{aligned}Y(z) = [H_0(z)G_0(z) + H_1(z)G_1(z)]X(z)\\ +[H_0(-z)G_0(z) + H_1(-z)G_1(z)]X(-z).\end{aligned} \qquad (1)$$

The perfect reconstruction condition (PRC) is given by the following two equations:

$$H_0(z)G_0(z) + H_1(z)G_1(z) = z^{-n_o}, \qquad (2)$$

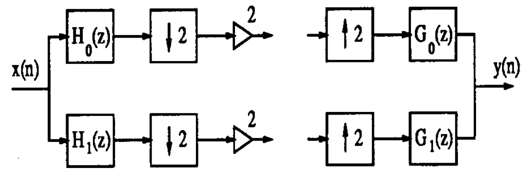

Figure 1: Two-channel analysis/synthesis filter bank.

and

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0 \qquad (3)$$

where $n_o \in \mathcal{Z}$, the set of integers. When the PRC is met, the reconstructed signal is then the exact replica of the input signal except for a delay, i.e., $y(n) = x(n - n_o)$, and the system is known as a PR system.

2.1. A Class of PR Systems

We note that (3) is satisfied if we choose

$$\begin{aligned}G_0(z) = H_1(-z),\\ G_1(z) = -H_0(-z).\end{aligned} \qquad (4)$$

With this choice, the PRC is met if

$$H_0(z)G_0(z) - \tfrac{1}{2}z^{n_o} \in \mathcal{T} \qquad (5)$$

for some $n_o \in \mathcal{Z}_o$, the set of odd integers. Here $\mathcal{T}$ is a set defined as

$$\mathcal{T} = \{E(z^2) : E(z^2) = \sum_{n=-\infty}^{\infty} e(2n)z^{-2n}\}. \qquad (6)$$

The product filter $H_0(z)G_0(z)$ in (5) belongs to a special class of filters known as halfband filters. We say that $H(z)$ is a halfband filter if there exist $k \in \mathcal{Z}$ and $n_o \in \mathcal{Z}_o$ which satisfy $z^kH(z) - \frac{1}{2}z^{-n_o} \in \mathcal{T}$ [14].

Suppose now that $G_0(z)$ can be expressed as

$$G_0(z) = G_D(z) + G_I(z), \qquad (7)$$

where $G_D(z)$ satisfies

$$H_0(z)G_D(z) - \tfrac{1}{2}z^{n_o} \in \mathcal{T}. \qquad (8)$$

Then (2) is satisfied if $H_0(z)G_I(z) \in \mathcal{T}$. It can easily be seen that $H_0(z)G_I(z) \in \mathcal{T}$ if $G_I(z)$ is in the form of $G_I(z) = B(z^2)H_0(-z)$. Thus by choosing $G_0(z)$ to be

$$G_0(z) = G_D(z) + B(z^2)H_0(-z), \qquad (9)$$

we can obtain a PR system regardless of the coefficient values of $B(z^2)$ as long as (8) is satisfied. Since the PRC depends only on $G_D(z)$ and is independent of $G_I(z)$, the design can be done in such a way that a PR system is first obtained with mild constraints on the spectral shape of $G_D(z)$, then the response of $G_0(z)$ is further improved by a proper choice of $B(z^2)$.

Halfband Pair Filter Bank (HBPF)

A special case of the above system, where $H_0(z)$ as well as $H_0(z)G_0(z)$ is a halfband filter, is shown below. This structure turn out to be very attractive in some applications. We choose $H_0(z)$ to be a halfband filter of the following form:

$$H_0(z) = \tfrac{1}{2} + zA(z^2). \tag{10}$$

Then $G_D(z)$ which satisfies the PRC is just a pure delay term z^{-n_o}, and $G_0(z)$ with $n_o = 1$ can be expressed as

$$G_0(z) = z^{-1}\left[1 + 2zB(z^2)\left(\tfrac{1}{2} - zA(z^2)\right)\right]. \tag{11}$$

When the frequency responses of $zA(z^2)$ and $zB(z^2)$ are real, the phase responses of $H_0(z)$ and $G_0(z)$ are linear, and the wavelets constructed from it have linear phase. Moreover, proper choices of these components lead to wavelets with the characteristics manifested in regular solutions. Some of the salient features of this structure are that the PRC is always satisfied regardless of the parameter values of $A(z^2)$ and $B(z^2)$ and that the PR system can be designed using known prototype filters without solving a set of linear equations. The HBPF was extended to the case of 2-D non-rectangular filter banks [15].

3. REGULAR WAVELETS

The condition under which a solution is regular is that a filter from which wavelets are constructed has enough number of zeros at $z = -1$. The sufficient condition for regularity is described in [2].

We construct linear-phase wavelets based on the HBPF. Since $G_0(z)$ in (11) is dependent of $H_0(z)$ in (10) in, the multiplicity of zeros of $G_0(z)$ clearly depends on the multiplicities of zeros of $H_0(z)$. The relationship between the multiplicities of zeros of $H_0(z)$ and $G_0(z)$ is given below.

Consider the subfilters $zA(z^2)$ and $zB(z^2)$ in (10) and (11). We constrain the frequency responses of $zA(z^2)$ and $zB(z^2)$ to be real for $\omega \in [0, 2\pi]$ and $-\frac{1}{2}$ at $\omega = \pi$. This ensures that (i) $H_0(z)$ and $G_0(z)$ have linear phase responses, (ii) $H_0(z) = G_0(z) = 1$ at $z = 1$, and (iii) $H_0(z)$ and $G_0(z)$ have at least one zero at $z = -1$.

Suppose $H_0(z)$ have m_h zeros at $z = -1$, i.e.,

$$H_0(z) = \left(\frac{1+z}{2}\right)^{m_h} F_h(z). \tag{12}$$

Since the frequency response of $zB(z^2)$ is constrained to be $-\frac{1}{2}$ at $z = -1$, we can establish the following:

$$D(z) = \tfrac{1}{2} + zB(z^2) = \left(\frac{1+z}{2}\right)^{m_d} F_d(z) \tag{13}$$

where m_d is the number of zeros of $D(z)$ at $z = -1$. Now assume that the values of $F_h(z)$ and $F_d(z)$ at $z = -1$ are finite positive real numbers. From (10), (11), (12), and (13), we can show that the number of zeros of $G_0(z)$ at $z = -1$ is given by

$$m_g = \min[m_h, m_d], \tag{14}$$

and we can express $G_0(z)$ as

$$G_0(z) = \left(\frac{1+z}{2}\right)^{m_g} F_g(z). \tag{15}$$

The fact that $F_h(z) = 1$ and $F_g(z) = 1$ at $z = 1$ follows from the constraints imposed on $zA(z^2)$ and $zB(z^2)$. When $H_0(z)$ and $G_0(z)$ satisfy the underlying assumptions made in deriving the regularity condition, regularity can be tested according to the criterion given in [2].

From (12) and (15), we can make the wavelet corresponding to the analysis section more regular at the expense of the one corresponding to the synthesis section (or vice versa), without losing the PR property, by exchanging the $\left(\frac{1+z}{2}\right)$ terms between $H_0(z)$ and $G_0(z)$. This is easily done by choosing a new set of lowpass filters $H_0'(z)$ and $G_0'(z)$ as

$$\begin{aligned} H_0'(z) &= \left(\tfrac{1+z}{2}\right)^{m_h - m_r} F_h(z), \\ G_0'(z) &= \left(\tfrac{1+z}{2}\right)^{m_g + m_r} F_g(z). \end{aligned} \tag{16}$$

In order for $H_0'(z)$ and $G_0'(z)$ to have at least one zeros at $z = -1$, the range of m_r should be chosen such that $-m_g < m_r < m_h$. Here $H_0'(z)$ is not a halfband filter unless $m_r = 0$. The proper choice of m_r may depend on the desirable spectral shapes of lowpass filters and the degree of regularity.

4. FIR SOLUTIONS

A special case of linear phase FIR halfband filters can be obtained by choosing the filter coefficients according to the Lagrange interpolation formula [12], and the transfer function of a Lagrange halfband filter can be expressed as

$$H(z) = \tfrac{1}{2} + zC(K, z^2) \tag{17}$$

where

$$C(K, z^2) = \sum_{n=1}^{K} c[K,n](z^{-2n} + z^{2n-2}) \tag{18}$$

and

$$c[K,n] = \frac{(-1)^{n+K-1}\prod_{i=1}^{2K}(K + \frac{1}{2} - i)}{(K-n)!\,(K-1+n)!\,(2n-1)}. \tag{19}$$

Here the index K is used to indicate that the filter has duration $4K - 1$. In [16] it was shown that a proper combination of factors of a Lagrange halfband filter could be chosen to get linear phase filters with simple coefficients. With imposition of regularity, the partition of a Lagrange halfband filter also provides solutions to the design problem [8] where a halfband filter with the largest number of zeros at $z = -1$ are desired. The difference in our approach is that we use a pair of Lagrange halfband filters to derive a set of filters, and no factorization is needed.

When $A(z^2)$ and $B(z^2)$ are chosen as

$$\begin{aligned} A(z^2) &= C(K_a, z^2) \\ B(z^2) &= C(K_b, z^2), \end{aligned} \tag{20}$$

$H_0(z)$ and $G_0(z)$ have multiple zeros at $z = -1$ with the multiplicities of $m_h = 2K_a$ and $m_g = \min[2K_a, 2K_b]$, respectively. For any $K_a \geq 1$, $H_0(z)$ satisfies the regularity condition. In the case of the synthesis filter $G_0(z)$, various examples that we tried showed that the iterated piecewise constant function as defined in [2] converged to a continuous function for $K_a, K_b \geq 2$. In Figure 2 we plot the scaling functions $\phi_h(t)$, $\phi_g(t)$ and the wavelets $\psi_h(t)$, $\psi_g(t)$ constructed with $K_a = K_b = 4$. Here the subscripts h and g are used to indicate that they correspond to the analysis section and the synthesis section, respectively. Fourier transforms of these scaling functions and wavelets are also shown in Figure 2.

5. IIR SOLUTIONS

An efficient sampling rate alteration scheme based on allpass sections was proposed in [13]. Our interest here is one particular form of this structure, namely, the Butterworth halfband filter. A Butterworth halfband filter of order N, where N is an odd integer, has N zeros at $z = -1$. This fact makes a Butterworth halfband filter a good choice for constructing smooth scaling functions and wavelets. Phase linearity, however, can not be achieved, as is the case with all causal IIR filters with rational transfer functions. When non-causal filtering is allowed, a zero-phase response can be obtained by cascading a causal filter with its anti-causal version.

Define $H'(z)$ to be $H(z)H(z^{-1})$ where $H(z)$ is a Butterworth halfband filter of order $N = 2K + 1$. The new filter $H'(z)$ is also a halfband filter and has a linear phase response. The transfer function of $H'(z)$ can be expressed as

$$H'(z) = \tfrac{1}{2} + zU(K, z^2) \tag{21}$$

where

$$U(K, z^2) = \tfrac{1}{2}\left(W(K, z^2) + z^{-2}W(K, z^{-2})\right) \tag{22}$$

and

$$W(K, z^2) = \frac{1}{2}\prod_{n=1}^{K} \frac{z^2 \cot^2 \frac{n\pi}{2K+1} + 1}{z^2 + \cot^2 \frac{n\pi}{2K+1}}. \tag{23}$$

With the choice of

$$\begin{aligned} A(z^2) &= U(K_a, z^2) \\ B(z^2) &= U(K_b, z^2), \end{aligned} \tag{24}$$

we have $m_h = 4K_a + 2$ and $m_g = \min[4K_a + 2, 4K_b + 2]$. The scaling functions and the wavelets constructed with $K_a = K_b = 1$ and corresponding Fourier transforms are shown in Figure 3.

6. CONCLUSIONS

In the paper we showed that linear-phase regular wavelets could be constructed using the HBPF with suitably chosen components. The HBPF always satisfies the PR property. The filter components which lead to regular solutions are obtained from a pair of Lagrange halfband filters for the non-recursive case. For the non-causal recursive case, we showed that the HBPF derived from Butterworth halfband filters resulted in smooth wavelets with linear-phase. The hybrid structure where one of the components $zA(z^2)$ and $zB(z^2)$ is derived from Lagrange halfband filters and the other from Butterworth halfband filters can also be used for the same application. In the non-recursive case the filter coefficients are simple and can be implemented with shift and add operations. Some of the examples of wavelets were shown.

References

[1] P. J. Burt and E. H. Adelson, "The laplacian pyramid as a compact image code," *IEEE Trans. Commun.*, vol. COM-31, pp. 532-540, April 1983.

[2] I. Daubechies, "Orthogonal bases of compactly supported wavelets," *Commun. on Pure and Applied Mathematics*, vol. XLI, pp. 909-996, 1988.

[3] S. Mallat, "A theory for multiresolution signal decomposition: the wavelet representation," *IEEE Trans. on PAMI*, vol. 11, no. 7, pp. 674-693, 1989.

[4] J. W. Woods, Ed., *Subband image coding*, Kluwer Academic Publishers, Norwell, MA, 1991.

[5] O. Rioul and M. Vetterli, "Wavelets and signal processing," *IEEE Sig. Proc. Mag.*, pp. 14-38, Oct. 1991.

[6] A. Cohen, I. Daubechies and J. C. Feauveau, "Biorthogonal bases of compactly supported wavelets," *AT&T Bell Lab. Tech. Report*, 1990.

[7] M. Vetterli and C. Herley, "Wavelets and filter banks: relationships and new results," *Proc. 1990 IASSP*, Albuquerque, NM, pp. 1723-1726, Apr. 3-6, 1990.

[8] M. Vetterli and C. Herley, "Wavelets and filter banks: Theory and design," to appear in *IEEE Trans. on Signal Proc.*, 1992.

[9] M. J. T. Smith and T. P. Barnwell, "Exact reconstruction technique for tree-structured subband coders," *IEEE Trans. on ASSP*, vol. ASSP-34, no. 3, pp. 434-441, June 1986.

[10] P. P. Vaidyanathan and P.-Q. Hoang, "Lattice structures for optimal design and robust implementation of two-band perfect reconstruction QMF banks" *IEEE Trans. on ASSP*, vol. ASSP-36, pp. 81-94, Jan. 1988.

[11] C. W. Kim and R. Ansari, "FIR/IIR Exact Reconstruction Filter Banks with Applications to Subband Coding of Images," *Proc. 1991 Midwest Circuits and Syst. Symp.*, Monterey, CA, May 1991.

[12] R. W. Schafer and L. R. Rabiner, "A Digital Signal Processing Approach to Interpolation," *Proc. of IEEE*, vol. 61, pp. 692-702, 1973.

[13] R. Ansari and B. Liu, "Efficient sampling rate alteration using recursive (IIR) digital filters," *IEEE Trans. on ASSP*, vol. ASSP-31, no. 6, pp. 1366-1373, Dec. 1983.

[14] M. G. Bellanger, J. L. Daguet and G. P. Lepagnol, "Interpolation, extrapolation and reduction of computation speed in digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-22, no. 4, pp. 231-235, August 1974.

[15] C. W. Kim and R. Ansari, "Subband decomposition procedure for quincunx sampling grids," *Proc. SPIE Conf. Visual Commun. and Image Processing*, vol. 1605, pp. 112-123, Boston, MA, Nov. 1991.

[16] R. Ansari, C. Guillemot and J. F. Kaiser, "Wavelet construction using Lagrange halfband filters" *IEEE Trans. on CAS*, vol. 38, no. 9, pp. 1116-1118, Sep. 1991.

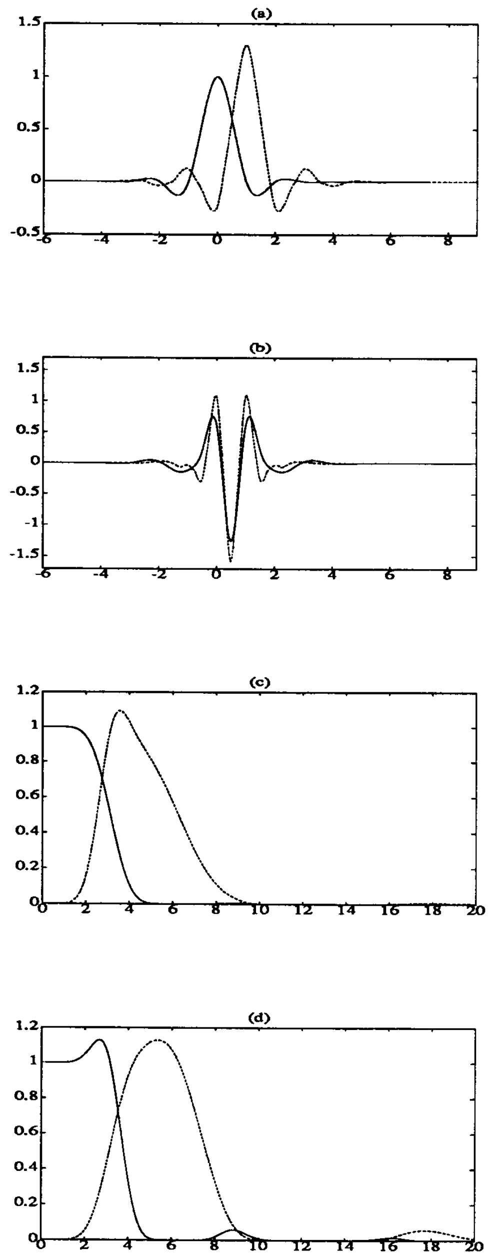

Figure 2: Examples of an FIR solution with $K_a, K_b = 4$ (a) scaling functions: $\phi_h(t)$ - solid; $\phi_g(t)$ - dotted, (b) wavelets: $\psi_h(t)$ - solid; $\psi_g(t)$ - dotted, (c) modulus of Fourier transforms $\hat{\phi}_h(\omega)$ and $\hat{\psi}_h(\omega)$, (d) modulus of Fourier transforms $\hat{\phi}_g(\omega)$ and $\hat{\psi}_g(\omega)$.

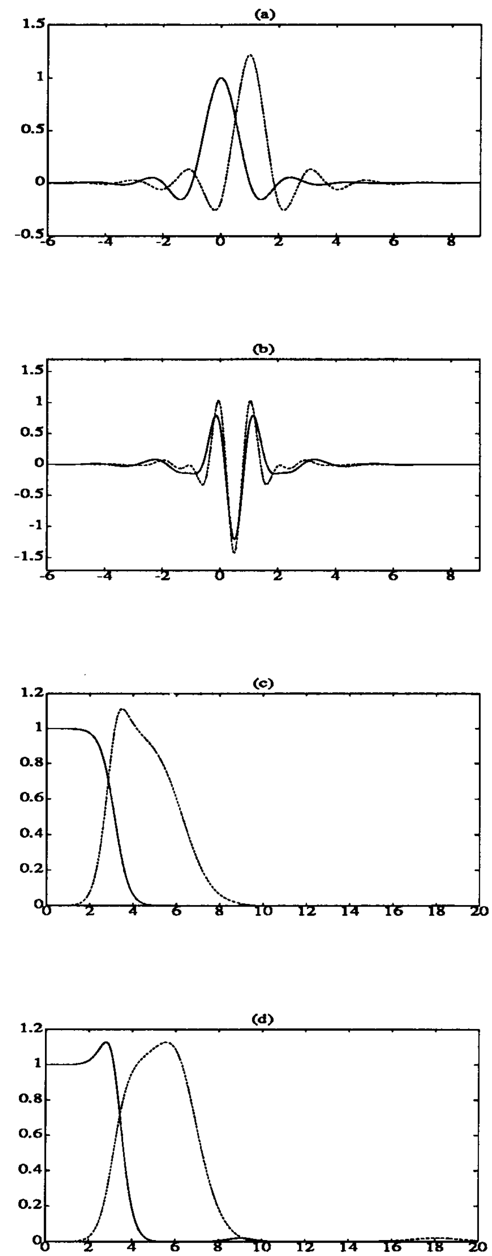

Figure 3: Examples of an IIR solution with $K_a, K_b = 1$ (a) scaling functions: $\phi_h(t)$ - solid; $\phi_g(t)$ - dotted, (b) wavelets: $\psi_h(t)$ - solid; $\psi_g(t)$ -dotted, (c) modulus of Fourier transforms $\hat{\phi}_h(\omega)$ and $\hat{\psi}_h(\omega)$, (d) modulus of Fourier transforms $\hat{\phi}_g(\omega)$ and $\hat{\psi}_g(\omega)$.