Introduction to Wavelet Transform

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Overview of Wavelet Course

- Sampling theorem and multirate signal processing
- Wavelets form an orthonormal basis of $L^2(R)$
- Time-frequency properties of wavelets and scaling functions
- Perfect reconstruction filterbanks for multirate signal processing and wavelets
- Lifting filterbanks
- Adaptive and nonlinear filterbanks in a lifting structure
- Frames, Matching Pursuit, Curvelets, EMD, ...
- Applications
Wavelets form an orthonormal basis of $L^2$:

Let $x(t) \in L^2$

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} 2^{k/2} w_{k,l} \psi(2^k t - l)$$

where

- $\psi(\cdot)$: wavelet basis function

- Wavelet (transform) coefficients: $w_{k,l}$

- Countable set of coefficients: $k,l$ are integers

- There are many wavelets satisfying the above equation
Wavelet coefficients

\[ w_{k,l} = < x(t), 2^{k/2} \psi(2^k t - l) > = \int_{-\infty}^{\infty} 2^{k/2} x(t) \psi(2^k t - l) \, dt \]

- Mother wavelet \( \psi(t) \) may have a compact support, i.e., it may be finite-extent \( \Rightarrow \) wavelet coefficients have temporal information
- The basis functions are constructed from the mother wavelet by translation and dilation
- Countable basis functions: 
  \[ 2^{k/2} \psi(2^k t - l), \; k, l \text{ are integers} \]
- Wavelets are orthonormal to each other
- Wavelet is a “bandpass” function
- In practice, we don't compute the above integral!
Multiresolution Framework

Let $w(t)$ be a mother wavelet:

$k = -1$

$k = 0$

$k = 1$
Fourier Transform (FT)

- Inverse Fourier Transform

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega \]

- \( e^{j\omega t} \) does not have a compact support, i.e., it is of infinite extent: \(-\infty < t < \infty \Rightarrow\) no temporal info

- \( e^{j\omega t} \) is also a bandpass function \( \Rightarrow \) delta at \( \omega \)

- \( F(\omega) \) is a continuous function (uncountable) of \( \omega \)

\[ \hat{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \]

- Uncountablity \( \Rightarrow \) integral in FT instead of summation in WT
Example: Haar Wavelet

\[ \psi(t) = \begin{cases} 
\frac{1}{2}, & \text{if } 0 \leq t \leq \frac{1}{2} \\
-\frac{1}{2}, & \text{if } \frac{1}{2} < t < 1 \\
0, & \text{otherwise}
\end{cases} \]

Corresponding scaling function:

\[ \phi(t) = \begin{cases} 
1, & \text{if } 0 \leq t < 1 \\
0, & \text{otherwise}
\end{cases} \]

- Haar wavelet is the only orthonormal wavelet with an analytic form
- It is not a good wavelet!
Wavelet and Scaling Function Pairs

- It is possible to have “zillions” of orthogonal mother wavelet functions
- It is possible to define a corresponding scaling function $\phi(t)$ for each wavelet
- Scaling function is a low-pass filter and it is orthogonal to the mother wavelet $\psi(t) \perp \phi(t)$
- Scaling coefficients (low-pass filtered signal samples):

$$c_{l,k} = \int_{-\infty}^{\infty} x(t) \phi(2^k t - l) dt$$
Wavelet and Scaling Function Properties-II

- Scaling function $\varphi(t)$ is not orthogonal to $\varphi(kt)$
- Wavelet $\psi(t)$ is orthogonal to $\psi(kt)$, for all integer $k$
- Haar wavelet:
  \[
  \varphi(t) = \frac{1}{\sqrt{2}} \varphi(2t) + \frac{1}{\sqrt{2}} \varphi(2t-1)
  \]
  \[
  \psi(t) = \frac{1}{\sqrt{2}} \varphi(2t) - \frac{1}{\sqrt{2}} \varphi(2t-1)
  \]
- Haar transform matrix: \[
  \begin{bmatrix}
  1 & 1 \\
  1 & -1
  \end{bmatrix}
  \]
- Daubechies 4\textsuperscript{th} order wavelet:
  \[
  \psi(t) = \frac{1}{4\sqrt{2}} \left[ (1-\sqrt{3}) \varphi(2t) - (3-\sqrt{3}) \varphi(2t-1) + (3+\sqrt{3}) \varphi(2t-2) - (1+\sqrt{3}) \varphi(2t-3) \right]
  \]
  \[
  \varphi(t) = \frac{1}{4\sqrt{2}} \left[ (1+\sqrt{3}) \varphi(2t) + (3+\sqrt{3}) \varphi(2t-1) + (3-\sqrt{3}) \varphi(2t-2) + (1-\sqrt{3}) \varphi(2t-3) \right]
  \]
Wavelet family (... $\psi(t/2), \psi(t), \psi(2t), \psi(4t),...$) covers the entire freq. band

- Ideal passband of $\psi(t)$: $[\pi, 2\pi]$
- Ideal passband of $\psi(2t)$: $[2\pi, 4\pi]$
- Almost no overlaps in frequency domain:

- Scaling function is a low-pass function:
  - Ideal passband of $\phi(t)$: $[0, \pi]$
  - Ideal passband of $\phi(2t)$: $[0, 2\pi]$
  - Scaling coefficients: low-pass filtered signal samples of $x(t)$:

\[
c_{l,k} = \int_{-\infty}^{\infty} x(t) \phi(2^k t - l) dt
\]
Daubechies 4 (D4) wavelet and the corresponding scaling function

- D4 and D12 plots:

- Wavelets and scaling functions get smoother as the number of filter coefficients increase
- D2 is Haar wavelet
**Multiresolution Subspaces of** \( L^2(R) \):

\[
V_{-1} = \text{span} \ \{ \phi(t/2 - l), \ l \ \text{integer} \} \\
V_0 = \text{span} \ \{ \phi(t - l), \ l \ \text{integer} \} \\
V_1 = \text{span} \ \{ \phi(2t - l), \ l \ \text{integer} \}
\]

A scale of subspaces:

\( \{0\} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(R) \)

- An ordinary analog signal may have components in all of the above subspaces:
  \[
c_{l,k} = \int_{-\infty}^{\infty} x(t) \phi(2^k t - l) dt \neq 0 \ \text{for all} \ k
\]
- A band limited signal will have \( c_{l,k} = 0 \ \text{for} \ k > K \)
Properties of multiresolution subspaces $V_j$

Multiresolution Decomposition of

The subspaces $V_j$ satisfy:

1) $V_j \subset V_{j+1}$ and $\bigcap_j V_j = \{0\}$ and $\bigcup_j V_j = \mathbb{L}^2$

2) Scale invariance: $f(t) \in V_j \iff f(2t) \in V_{j+1}$

3) Shift invariance: $f(t) \in V_0 \iff f(t-k) \in V_0$
Wavelet subspaces

- \( W_0 = \text{span}\{ \psi(t-l), \text{integer } l \}, \ldots \)

- \( W_j \) does not contain \( W_k, j > k \) (but \( V_j \) does contain \( V_k \))

- It is desirable to have \( V_j \) to be orthogonal to \( W_j \)
Geometric structure of subspaces

- $W_{j+1}$ is the “z-axis”, $V_{j+2}$ is the 3-D space ...

Orthogonal choice:

\[ V_{j+1} = V_j \oplus W_j \]

$V_j \cap W_j = \{0\}$ (This is not empty set but $V_j$ are differences between ...
Ideal frequency contents of wavelet and scaling subspaces:

- Subspace $V_0$ contains signals with freq. content $[0, \pi]$.
- Subspace $W_0$ contains signals with freq. content $[\pi, 2\pi]$.
- Subspace $V_1$ contains signals with freq. content $[0, 2\pi]$.
- Subspace $W_1$ contains signals with freq. content $[2\pi, 4\pi]$.
- Subspace $V_2$ contains signals with freq. content $[0, 4\pi]$.
Structure of subspaces:

- \[ V_1 = V_0 \oplus W_0 \]
- \[ V_2 = V_1 \oplus W_1 \quad \text{etc.} \]
- \[ V_2 = V_0 \oplus W_0 \oplus W_1 \]
- \[ V_3 = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \]
- \[ L^2(\mathbb{R}) = V_0 \oplus \bigoplus_{j=1}^{\infty} W_j \]

- Geometric analogy: each wavelet subspace adds another dimension
Projection of a signal onto a subspace \( V_0 \)

- Projection \( x_o(t) \) of a signal \( x(t) \) onto a subspace \( V_0 \) means: 1\(^{st}\) compute:

\[
c_{n,o} = \int_{-\infty}^{\infty} x(\tau) \phi(\tau - n) d\tau
\]

for all integer \( n \)

and form \( x_o(t) = \sum_n c_{n,o} \phi(t-n) \) which is a smooth approximation of the original signal \( x(t) \)

- This is equivalent to low-pass filtering \( x(t) \) with a filter with passband \([0, \pi]\) and sample output with \( T=1 \)

- As a result we don't compute the above integrals in practice:

\[
x_o(t) = \sum_n x_o[n] \phi(t-n)
\]
Regular sampling: \( f_{lp}(t) = \sum f_{lp}[n] \text{sinc}(t-n) \)
Sampling-II

* The relation between $f_1[n]$ and $f_0[n]$. 

$f_1(t) = \sum_n f_1[n] \phi (2t-n)$ is a better approximation than $f_0(t)$. 

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$f_1(t) = \sum_n f_1[n] \phi (2t-n)$ is a better approximation than $f_0(t)$.
Projection onto the subspace $V_j$ (freq. content: $[0, 2^j \pi]$)

Given $f(t) \in L^2(\mathbb{R})$:

$$f_j(t) = P_j f(t)$$

$$V_j = \text{span} \{ 2^{j/2} \phi(2^j t - n), \ n \in \mathbb{Z} \}$$

$$f_j(t) = \sum_{n=-\infty}^{\infty} 2^{j/2} f_j[n] \phi(2^j t - n)$$

where $f_j[n] = \int_{-\infty}^{\infty} f(t) 2^{j/2} \phi(2^j t - n) \, dt$ =

This is almost equivalent to Shannon sampling with $T=1/2^j$
Wavelet Equation (Mallat)

- \( W_0 \subset V_1 \Rightarrow \)
  \[ \psi(t) = \sqrt{2} \sum_k d[k] \varphi(2t-k) \]

- \( d[k] = \sqrt{2} \langle \psi(t), \varphi(2t-k) \rangle \), \( \psi(t) = 2 \sum_k g[k] \varphi(2t-k) \)

- \( g[k] = \sqrt{2} d[k] \) is a discrete-time half-band high-pass filter

- Example: Haar wavelet
  \[ \psi(t) = \varphi(2t) - \varphi(2t-1) \Rightarrow d[0] = \sqrt{2}/2 \), \( d[1] = -\sqrt{2}/2 \)

- \( g \) and \( d \) are simple discrete-time high-pass filters
Scaling Equation

- Subspace $V_0$ is a subset of $V_1$ =>
  \[ \phi(t) = 2 \sum_k h[k] \phi(2t-k) \]

  where $h[k] = \sqrt{2} \ < \phi(t), \phi(2t-k) >$

- $h[k] = \sqrt{2} c[k]$ is a half-band discrete-time low-pass filter with passband: $[0, \pi/2]$

- In wavelet equation $g[k]$ is a high-pass filter with passband $[\pi/2, \pi]$
Fourier transforms of wavelet and scaling equations

\[ \hat{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(t) e^{-i\omega t} dt, \quad W(\omega) = \int_{-\infty}^{\infty} \psi(t) e^{-i\omega t} dt \]

\[ \phi(t) = 2 \sum h[k] \phi(2t - k) \Rightarrow \hat{\phi}(\omega) = H(e^{i\omega/2}) \hat{\phi}(\omega/2) \]

Similarly, \[ W(\omega) = G(e^{i\omega/2}) \hat{\phi}(\omega/2) \]

\[ \hat{\phi}(\omega) = H(e^{i\omega/2}) H(e^{i\omega/4}) H(e^{i\omega/8}) \ldots \]

Orthogonality Condition:

\[ |H(e^{i\omega})|^2 + |H(e^{i(\omega + \pi)})|^2 = 1. \]

\[ H(\pi) = 0 \quad (\text{zero at } \omega=\pi \text{ or } z=-1) \]

\( H(e^{i\omega}), G(e^{i\omega}) \) are the discrete-time Fourier transforms of \( h[k] \) & \( g[k] \), respectively.
Two-channel subband decomposition filter banks (Esteban & Galant 1975)

\[ |H(e^{i\omega})|^2 + |H(e^{i(\omega+\pi)})|^2 = 1 \quad \text{or} \quad |H(e^{i\omega})|^2 + |G(e^{i\omega})|^2 = 1 \]

Filterbanks in multirate signal processing: low-pass and high-pass filter the input discrete signal \( x[n] \) and downsample outputs by a factor of 2:

It is possible to reconstruct the original signal from subsignals using the synthesis filterbank.
Wavelet construction for Multiresolution analysis

- Start with a perfect reconstruction filter bank:

1) Filter bank $h[k]$ and $g[k]$
2) $\hat{\phi}(\omega) = \sum_{l=1}^{\infty} H(e^{i\frac{\omega}{2^l}})$ (convergence problems may occur!)
3) $\phi(t) = \mathcal{F}_c^{-1}\{\hat{\phi}(\omega)\}$ and $w(t) = \mathcal{F}_c^{-1}\{G(e^{i\omega})\hat{\phi}(\omega/2)\}$

- But we don't compute inner products with $\Psi(t)$ and $\phi(t)$ in practice!
- We only use the discrete-time filterbanks!
Filter Bank Design (Daubechies in 1988 but earliest examples in 1975)

Example half-band filters: Lagrange filters $p[n]$:

- $p[n] = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$, $p[n] = 2 \begin{bmatrix} -1/32 & 0 & 9/32 & 1 & 9/32 & 0 & -1/32 \end{bmatrix},...$
Mallat's Algorithm (≡ Signal analysis with perfect reconstruction filter banks)

You can obtain lower order approximation and wavelet coefficients from higher order approximation coefficients:

Reconstruction:

\[ x_{j+1}[k] = \sum_{l} c[l-2k] x_{j+1}[l] \]

\[ b_{j}[k] = \sum_{l} c[l-2k] x_{j+1}[l] \]

\( c[k] = h[k]/\sqrt{2} \) and \( d[k] = g[k]/\sqrt{2} \) are discrete-time low-pass and high-pass filters, respectively.
Mallat's Algorithm (≡ Signal analysis with perfect reconstruction filter banks)

You can obtain lower order approximation and wavelet coefficients from higher order approximation coefficients:

\[ x_j[k] = \sum_{\ell} c[\ell-2k] x_{j+1}[\ell] \]
\[ b_j[k] = \sum_{\ell} d[\ell-2k] x_{j+1}[\ell] \]

Reconstruction using the synthesis filterbank:

\[ c[k] = h[k]/\sqrt{2} \] and \[ d[k] = g[k]/\sqrt{2} \] are discrete-time low-pass and high-pass filters, respectively.
Mallat's algorithm (tree structure)

- Obtain $x_{j-1}[n]$ and wavelet coefficients $b_{j-1}[n]$ from $x_j[n]$
- Obtain $x_{j-2}[n]$ & wavelet coefficients $b_{j-2}[n]$ from $x_{j-1}[n]$
- Obtain $x_{j-3}[n]$ & wavelet coefficients $b_{j-3}[n]$ from $x_{j-2}[n]$

- Wavelet tree representation of $x_j[n]$:  
  $$x_j[n] \equiv \{ b_{j-1}[n], b_{j-1}[n],...,b_{j-N}[n]; x_{j-N}[n] \}$$
  where $b_{j-1}[n], b_{j-1}[n],...,b_{j-N}[n]$ are the wavelet coefficients at lower resolution levels
- Use a filterbank (e.g. Daubechies-4) to obtain the wavelet coefficients
Discrete-time Wavelet Transform

- Discrete-time filter-bank implementation:
  H is the low-pass and G is the high-pass filter of the wavelet transform

- Subband decomposition filterbank acts like a “butterfly” in FFT
- Perfect reconstruction of $x_j$ from subsignals, $x_{j-3}[n],..,b_{j-1}[n]$ is possible
- Both time and freq. information is available but Heisenberg's principle applies
Wavelet Packet Transform

Length of $x[n]$ is $N \Rightarrow$ Lengths of $v_0$, $v_1$, $v_2$, and $v_3$ are $N/4$
Two-dimensional filterbanks for image processing

filters:

\[ h_0[n] = h_q[n_1] h_q[n_2] \]
\[ h_1[n_1,n_2] = h_q[n_1] h_h[n_2] \]
\[ h_2[n_1] = h_h[n_1] h_q[n_2] \]
\[ h_3[n] = h_h[n_1] h_h[n_2] \]

Freq. domain picture
Example

- Cont. time signal $x(t) = 1$ for $t<5$ and $2$ for $t >5$
- Sample this signal with $T=1 \equiv$ Project it onto $V_0$ of Haar multiresolution decomposition using $h=\{\frac{1}{2} \ \frac{1}{2}\}$, $g=\{\frac{1}{2} \ -\frac{1}{2}\}$:
  - $x[n] = (... 1 1 1 1 1 2 2 2 2 2 2 2 ....)$
- Perform single level Haar wavelet transform:
  - Lowpass filtered signal: $(... 1 1 1 1 1.5 2 2 2 2...)$
  - Low-resolution subsignal: $(... 1 1 1.5 2 2...)$
  - Highpass filtered signal: $(... 0 0 0 0.5 0 0 0 0...)$
  - $1^{st}$ scale wavelet subsignal $(... 0 0 0.5 0 0 0...)$
- We can estimate the location of the jump from the nonzero value of the wavelet signal
- Haar is not a good wavelet transform because the wavelet signal of $x[n-1]$ would be $(...0 0 0 0 0 0...)$
Toy Example: signal data compression

- Original $x[n] = (1\ 1\ 1\ 1\ 2\ 2\ 2\ 2)$
- 8 bits/sample => $8 \times 8 = 64$ bits
- Single level Haar wavelet transform:
  Low-resolution subsignal: $(1\ 1\ 1.5\ 2\ 2)$
  $5 \times 8$ bits/pel $= 40$ bits
  1$^{st}$ scale wavelet signal: $(0\ 0\ 0.5\ 0\ 0)$
  Only store the nonzero value (9 bits) and its location (3 bits)
  Total # of bits to store the wavelet signals = 52 bits
- Since 52 bits < 64 bits it is better to store the wavelet sub-signals instead of the original signal
Denoising Example

- Original: \( x[n] = \ldots 1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ ... \) 
- Corrupted: \( x_c[n] = \ldots 1 \ 1.2 \ 1 \ 1 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ ... \) 
- Single level Haar wavelet transform of \( x_c[n] \) using \( h = \{ \sqrt{2}/2, \sqrt{2}/2 \} \), \( g = \{ \sqrt{2}/2, -\sqrt{2}/2 \} \):
  - Low-resolution subsignal \( x_l = \ldots 1.49 \ 1.59 \ 1.51 \ 2.828 \ 2.828 \ldots \) 
  - 1st scale wavelet signal: \( \ldots -0.15 \ -0.06 \ 0.354 \ 0 \ 0 \ldots \) 
  - Soft-thresholded wavelet signal: \( x_s = \ldots 0 \ 0 \ 0.354 \ 0 \ 0 \ldots \) 
- Restored signal from \( x_l \) and \( x_s \):
  \( x_r[n] = \ldots 1.1 \ 1.13 \ 1.04 \ 0.98 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ ... \) 
- Better denoising results can be obtained with higher order wavelets using longer filters which provide better smoothing of the low-resolution signal.
2-D image processing using a 1-D filterbank (separable filtering)
2-D image processing using a 1-D filter

Seperable processing in each channel of the 2-D filterbank:
2-D wavelet transform of an image

- Single scale decomposition:

- "low-low" subimage can be further decomposed to subimages.

Figure 1: Original frame and its single level wavelet subimages.
Image Compression

- JPEG-2000 (J2K) is based on wavelet transform
- Energy of the high-pass filtered subimages are much lower than the low-low subimage
- Most of the wavelet coefficients are close to zero except those corresponding to edges and texture
- Threshold low-valued wavelet coefficients to zero
- Take advantage of the correlation between wavelet coefficients at different resolutions
- JPEG and MPEG are still preferred because of local nature of DCT and Intellectual Property issues of J2K
Lifting (Sweldens)

- Filtering after downsampling:

- It reduces computational complexity
- It allows the use of nonlinear (Pesquet), binary and adaptive filters (Cetin) as well
Adaptive Lifting-II

- Reconstruction filterbank structure from Gerek and Cetin, 2000
The basic idea of lifting: If a pair of filters \( (h,g) \) is complementary, that is it allows for perfect reconstruction, then for every filter \( s \) the pair \( (h',g) \) with allows for perfect reconstruction, too.

- \( H'(z)=H(z)+s(z^2)G(z) \) or
- \( G'(z)=G(z)+s(z^2)H(z) \)

Of course, this is also true for every pair \( (h,g') \) of the form .

The converse is also true: If the filterbanks \( (h,g) \) and \( (h',g) \) allow for perfect reconstruction, then there is a unique filter \( s \) with .

http://pagesperso-orange.fr/polyvalens/clemens/lifting/lifting.html
Equations

\[ x \sim \sum_{n=-\infty}^{\infty} |H(e^{iw})|^2 + |H(e^{i(w+\pi)})|^2 = 1 \sim \text{or} \sim |H(e^{iw})|^2 + |G(e^{iw})|^2 = 1 \]

\[ \phi(t) = 2\sum h[k] \phi(2t-k) \Rightarrow \hat{\phi}(w) = H(e^{iw/2})\hat{\phi}(w/2) \]

\[ \hat{\phi}(w) = \int_{-\infty}^{\infty} \phi(t) e^{-iwt} dt, \quad W(w) = \int_{-\infty}^{\infty} \psi(t) e^{-iwt} dt \]