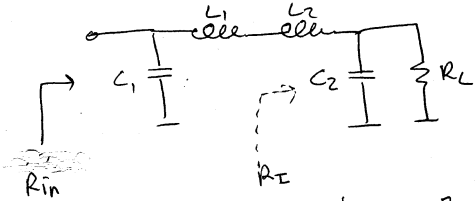
A note on π-match
Q calculation



let $Z_{C_1} = +jX_{C_1} = -j\frac{1}{w_o C_1}$ $\quad Z_L = jX_L = jw_oL$

$Z_{C_2} = jX_{C_2} = -j\frac{1}{w_o C_2}$

$Z_{L_1} = jX_{L_1} = jw_oL_1$

$Z_{L_2} = jX_{L_2} = jw_oL_2$

$$Q = Q_L + Q_R = \frac{w_o(L_1+L_2)}{R_I} = \frac{X_L}{R_I}$$

$$Q_L = w_o C_1 R_{in} = \frac{R_{in}}{-X_{C_1}}$$

$$Q_R = w_o C_2 R_L = \frac{R_L}{-X_{C_2}}$$

$(Q_L^2 + 1) R_I = R_{in}$ equ. 1

$(Q_R^2 + 1) R_I = R_L$ equ. 2

The equation for overall Q was:

$$Q = \frac{w_o(L_1+L_2)}{R_I} = Q_L + Q_R = \sqrt{\frac{R_{in}}{R_I} - 1} + \sqrt{\frac{R_L}{R_I} - 1}$$

This equation requires iteration to solve for R_I especially if the Q is low. But this system of

equations actually has an analytical solution.
Diving equ. 1 by equ. 2, we get:

$$\frac{R_{in}}{R_L} = \frac{Q_L^2+1}{Q_R^2+1} \quad , \quad \text{letting } K = \frac{R_{in}}{R_L}$$

$$K = \frac{Q_L^2+1}{Q_R^2+1} \qquad \text{equ. } 3$$

$$K Q_R^2 + K = Q_L^2 + 1$$

$$K Q_R^2 + K - 1 = Q_L^2 = (Q - Q_R)^2 = Q^2 - 2QQ_R + Q_R^2$$

$$K Q_R^2 - Q_R^2 + \underbrace{2Q}_{b} Q_R \underbrace{- Q^2 + K - 1}_{c} = 0$$

$K - 1 = a$

This is a quadratic equation, the solution of which is given by:

$$Q_{R_{1,2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2Q \pm \sqrt{4Q^2 - 4(K-1)(K-1-Q^2)}}{2(K-1)}$$

$$= \frac{\cancel{2}\sqrt{Q^2 - (K^2 - K - KQ^2 - K + 1 + Q^2)} - \cancel{2}Q}{\cancel{2}(K-1)}$$

$$Q_{R_{1,2}} = \frac{\sqrt{\cancel{Q^2} - K^2 + K + KQ^2 + K - 1 - \cancel{Q^2}} - Q}{K-1}$$

Rearranging the terms and taking only the positive signed $\sqrt{\ }$ term, we get

$$Q_R = \frac{\left[Q^2K + 2K - K^2 - 1\right]^{1/2} - Q}{K - 1} \qquad \underline{\text{Equ. } 4}$$

which lets us directly calculate Q_R hence Q_L. The other root of the equation is obviously negative. Note that the numerator of the equation changes sign along with the denominator at $K = 1$, so Q_R turns out to be positive even for $1 > K > 0$. For $K \leq 0$ we are talking about lossless loads and negative resistance which are beyond the scope of this note.

Extension to T-match

In T-match the dual equations are

$$Q = w_0 R_I (C_1 + C_2) = \sqrt{\frac{R_I}{R_{in}} - 1} + \sqrt{\frac{R_I}{R_L} - 1} = Q_L + Q_R$$

$$(Q_L^2 + 1) R_{in} = R_I \qquad \text{equ-} 5$$

$$(Q_R^2 + 1) R_L = R_I \qquad \text{equ-} 6$$

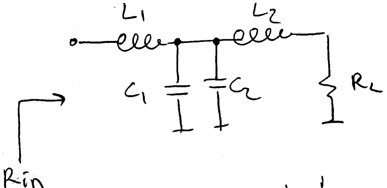
dividing equ 5 by equ-6, we obtain

$$\frac{Q_L^2 + 1}{Q_R^2 + 1} = \frac{R_L}{R_{in}} = K \qquad \text{equ-} 7$$

This time letting $K = \frac{R_L}{R_{in}}$ instead of $\frac{R_{in}}{R_L}$

The rest of the derivation is the same ⟹

$$Q_R = \frac{[Q^2 K + 2K - K^2 - 1]^{1/2} - Q}{K - 1}$$



$$Q_R = \frac{X_{L2}}{R_L}$$

$$Q_L = \frac{X_{L1}}{R_{in}}$$

$$Q = \frac{R_I}{X_C}$$

Of course, one must be carefull in substituting the appropriate variables in dual Q expressions.