Elmore Definitions — 14.10.2002 — (4)

Delay of systems in cascade

Delay of a circuit can be defined using moments of the impulse response

$$T_D \equiv \frac{\int_{-\infty}^{+\infty} t\,h(t)\,dt}{\int_{-\infty}^{+\infty} h(t)\,dt}$$

Elmore delay

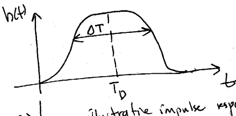
i.e. the time taken for the impulse to reach its center of mass.

But in the frequency domain it corresponds to

$$\int_{-\infty}^{+\infty} t\,h(t)\,dt = -\frac{1}{j2\pi}\frac{d}{df}H(f)\Big|_{f=0}$$

$$\int_{-\infty}^{+\infty} h(t)\,dt = H(0)$$

$\Rightarrow$



illustrative impulse response

$$T_D \equiv \frac{-1}{j2\pi\,H(0)}\frac{d}{df}H(f)\Big|_{f=0}$$

Using this definition it is easy to show that the elmore delay of a single-pole low-pass system to be τ

emphasize $H(f)$ is not the fourier transform, but it is a function f not $\omega = 2\pi f$, and actually

$$H(f) = \mathbf{H}(2\pi f)$$

frequency response — fourier transform

①

14.10.2005

$$\int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt = H(\omega) = H(f)$$
$$= H(2\pi f)$$

$$\cancel{\frac{d}{d\omega} H(\omega) = \frac{d}{d\omega} H(f) = \frac{d}{d\omega} H\left(\frac{\omega}{2\pi}\right)}$$

$$\frac{d}{d\omega}\int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt = -j \int (t f(t) e^{-j\omega t})\, dt = \frac{d}{d\omega} H(\omega) = \frac{1}{2\pi}\frac{d}{df} H(f) = \frac{1}{2\pi}$$

$$\omega = 2\pi f$$
$$\frac{d\omega}{df} = 2\pi$$
$$\frac{d}{df} H(f) = \frac{d\omega}{df}\cdot\frac{d}{d\omega} H(\omega) = 2\pi\, \frac{d H(\omega)}{d\omega}$$
$$\frac{d}{df} H(\omega) = \frac{d\omega}{df}\,\frac{dH(\omega)}{d\omega}$$

$$\int_{-\infty}^{\infty} t f(t)\, dt = \int_{-\infty}^{\infty} t f(t) e^{-j\omega t} dt \Big|_{W=0} = \frac{1}{-j}\frac{d}{d\omega} H(\omega)\Big|_{\omega=0}$$

$$\frac{d}{d\omega}\int_{-\infty}^{+\infty} t f(t)\, e^{-j\omega t}\, dt = \frac{1}{-j}\frac{d^2}{d\omega^2} H(\omega)$$

$$(-j)^2 \int_{-\infty}^{+\infty} t^2 f(t) e^{-j\omega t} dt = \frac{d^2}{d\omega^2} H(\omega)$$

$$\int_{-\infty}^{+\infty} t^2 f(t)\, dt = \int_{-\infty}^{+\infty} t^2 f(t) e^{-j\omega t}\, dt \Big|_{W=0} = -\frac{d^2}{d\omega^2} H(\omega)\Big|_{\omega=0} = -\frac{d^2}{d\omega^2} H(0)$$

$$T_D \equiv \frac{\int_{-\infty}^{+\infty} t f(t)\, dt}{\int_{-\infty}^{+\infty} f(t)\, dt} = \frac{\frac{1}{-j}\frac{d}{d\omega} H(\omega)\Big|_{\omega=0}}{H(0)}$$

≡ Elmore delay

②

14-10-2005

If we have 2 systems in cascade with $h_1(t)$ & $h_2(t)$ and corresponding fourier transforms

$$T_{D,\,total} \equiv \frac{1}{-j H_1(0) H_2(0)} \frac{d}{dw} H_1(w) H_2(w) \Big|_{w=0}$$

$$= \frac{1}{-j H_1(0) H_2(0)} \left[H_1(w) \frac{d}{dw} H_2(w) + H_2(w) \frac{d}{dw} H_1(w) \right] \Big|_{w=0}$$

$$= \frac{\frac{d}{dw} H_2(w) \big|_{w=0}}{-j H_2(0)} + \frac{\frac{d}{dw} H_1(w) \big|_{w=0}}{-j H_1(0)} = T_{D_1} + T_{D_2}$$

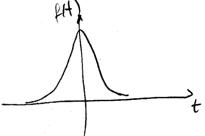
2-A
14-10-2005

rise time.

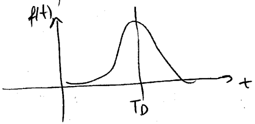
The rise time of an impulse response centered around $t=0$ can be defined as

$$\left(\frac{\Delta t}{2}\right)^2 = \frac{\int_{-\infty}^{+\infty} t^2 (f(t))\,dt}{\int_{-\infty}^{+\infty} f(t)\,dt}$$

which is a measure of the spread i.e. radius of gyration



where $T_D = 0$



if $T_D \neq 0$

it should be moved to T_D by

$$\left(\frac{\Delta T}{2}\right)^2 = \frac{\int_{-\infty}^{+\infty} (t-T_D)^2 f(t)\,dt}{\int_{-\infty}^{+\infty} f(t)\,dt} = \frac{\int_{-\infty}^{+\infty} t^2 f(t)\,dt}{\int_{-\infty}^{+\infty} f(t)\,dt} - T_D^2$$

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$$\int_{-\infty}^{+\infty} t^2 h(t) = -\frac{1}{(2\pi)^2} \frac{d^2}{df^2} H(f)\Big|_{f=0}$$

$$\frac{(\Delta T)^2}{4} = \frac{\int_{-\infty}^{+\infty} t^2 h(t)\,dt}{\int_{-\infty}^{+\infty} h(t)\,dt} - T_D^2 = \frac{\int t^2 h(t)\,dt}{\int h(t)\,dt} - \left[\frac{\int t\,h(t)\,dt}{\int h(t)\,dt}\right]^2$$

$$\frac{(\Delta T)^2}{4} = \frac{-\frac{1}{(2\pi)^2}\ddot{H}}{H} - \left[-\frac{1}{(j2\pi)}\frac{\dot{H}}{H}\right]^2 = -\frac{\frac{\ddot{H}}{(2\pi)^2}}{H} - \left[\frac{1}{(2\pi j)^2}\frac{\dot{H}^2}{H^2}\right]$$

(formula 124 sign error at the back)

$$= -\frac{\ddot{H}}{(2\pi)^2 H} + \frac{1}{(2\pi)^2}\frac{\dot{H}^2}{H^2} = \frac{1}{(2\pi)^2 H}\left[-\ddot{H} + \frac{\dot{H}^2}{H}\right]$$

$$\frac{\Delta T^2}{4} = \frac{-\ddot{H}H + \dot{H}^2}{H^2(2\pi)^2} \Rightarrow \boxed{\pi^2 \Delta T^2 = \frac{-\ddot{H}H + \dot{H}^2}{H^2}}$$

$$\pi^2 \Delta T^2 = \frac{-H[\ddot{H}_1 H_2 + 2\dot{H}_1\dot{H}_2 + H_1\ddot{H}_2] + [\dot{H}_1 H_2 + H_1\dot{H}_2]^2}{H^2}$$

$$\pi^2 \Delta T^2 = \frac{-H_1 H_2 \ddot{H}_1 H_2 - 2H_1H_2\dot{H}_1\dot{H}_2 - H_1H_2H_1\ddot{H}_2 + \dot{H}_1^2 H_2^2 + 2H_1\dot{H}_1H_2\dot{H}_2 + H_1^2\dot{H}_2^2}{H_1^2H_2^2}$$

$$= \frac{-H_1\ddot{H}_1H_2^2 \cancel{- 2H_1H_2\dot{H}_1\dot{H}_2} - H_1^2H_2\ddot{H}_2 + \dot{H}_1^2H_2^2 \cancel{+ 2H_1H_2\dot{H}_1\dot{H}_2} + H_1^2\dot{H}_2^2}{H_1^2H_2^2}$$

$$= \frac{-H_1\ddot{H}_1}{H_1^2} + \frac{\dot{H}_1^2}{H_1^2} + \frac{-H_2\ddot{H}_2 + \dot{H}_2^2}{H_2^2} = \pi^2\Delta T_1^2 + \pi^2\Delta T_2^2$$

$$\Rightarrow \Delta T^2 = \Delta T_1^2 + \Delta T_2^2$$

$$H = H_1 H_2$$

$$\frac{d}{dt} H = \frac{d}{dt} H_1 H_2 = \dot{H}_1 H_2 + H_1 \dot{H}_2$$

$$\frac{d}{dt}\frac{d}{dt} H = \frac{d}{dt}\frac{d}{dt} H_1 H_2 = \frac{d}{dt}\left(\dot{H}_1 H_2 + H_1 \dot{H}_2\right)$$

$$\frac{d^2}{dt^2} H_1 H_2 = \ddot{H}_1 H_2 + \dot{H}_1 \dot{H}_2 + \dot{H}_1 \dot{H}_2 + H_1 \ddot{H}_2$$

$$\pi^2 \Delta T_1^2 = \frac{-\ddot{H}_1 H_1 + \dot{H}_1^2}{H_1^2} \qquad \pi^2 \Delta T_2^2 = \frac{-\ddot{H}_2 H_2 + \dot{H}_2^2}{H_2^2}$$

LINEAR PHASE CIRCUIT

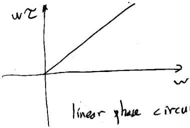
$$H(w) = \int_{-\infty}^{+\infty} h(t)\, e^{-jwt}\, dt$$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(w)\, e^{jwt}\, dw$$

$$h(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(w)\, e^{jw(t-\tau)}\, dw$$

$$h(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(w)\, e^{jwt} \cdot e^{-jw\tau}\, dw$$

$$h(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(w) e^{-jw\tau}\, e^{jwt}\, dw$$



linear phase circuit

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weighted average of $(t-T_D)^2 = d^2$

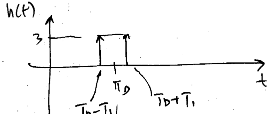
total mass $= \int_{-\infty}^{+\infty} h(t)dt$

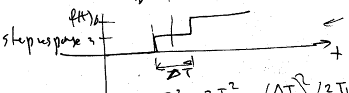
the tighter the distance $\Rightarrow$ the shorter is the rise time

$$\left(\frac{\Delta T}{2}\right)^2 = \frac{\int_{-\infty}^{+\infty}(t-T_D)^2 h(t)dt}{\int_{-\infty}^{+\infty} h(t)dt} = \frac{\int t^2 h(t)dt}{\int h(t)dt} - 2T_D \underbrace{\frac{\int t\,h(t)dt}{\int h(t)dt}}_{=T_D} + \frac{T_D^2 \cancel{\int h(t)dt}}{\cancel{\int h(t)dt}}$$

$$\left(\frac{\Delta T}{2}\right)^2 = \frac{\int t^2 h(t)dt}{\int h(t)dt} - \cancel{2}T_D^2 + T_D^{\cancel{2}} = \frac{\int_{-\infty}^{+\infty} h(t)dt}{\int h(t)dt}$$

Example





← rise time $= 2T_1$

$$T_1^2 = \frac{3T_1^2 + 3T_1^2}{6} = \left(\frac{\Delta T}{2}\right)^2 = \left(\frac{2T_1}{2}\right)^2 = T_1^2$$