DFT:

$$X[k] \triangleq \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N} n}, \quad k=0,1,\dots,N-1.$$

where $x[n]$ is a finite-extent signal.

(Inverse DFT: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k}{N} n}$, $n=0,1,\dots,N-1$

- Finite sum.
- $X[k]$ is computed at discrete-indices. $k=0,1,\dots,N-1$.

$\therefore$ DFT can be computed using ordinary computer

* Let $x[n]$ be a finite-extent signal s.t. $x[n]=0$ for $n<0$ and $n \geq L (\leq N-1)$.

D.T.F.T.

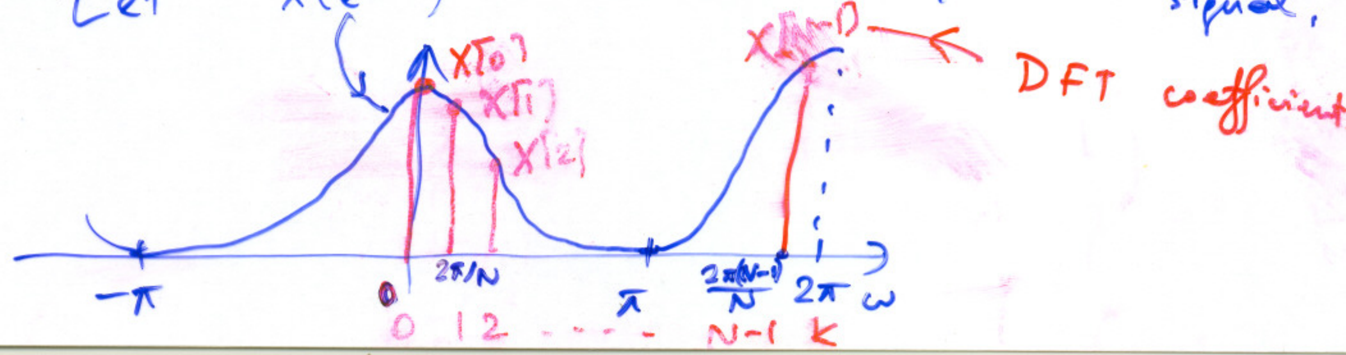
$$X(e^{j\omega}) = \sum_{n=0}^{L-1} x[n] e^{-j\omega n}, \quad \omega \text{ is a cont. variable.}$$

D.F.T.

$$X[k] = \sum_{n=0}^{L-1} x[n]\, e^{-j\frac{2\pi k}{N} n}, \quad k=0,1,\dots,N-1.$$

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}, \quad k=0,1,\dots,N-1.$$

Ex// Let $X(e^{j\omega})$ be the D.T.F.T. of a finite-extent signal,



* DFT has a period of N. (when we compute DFT outside the range $k=0,1,\ldots,N-1$).
because D.T.F.T is 2π periodic

$$X[N] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi \cancel{N}}{\cancel{N}}n} = \sum_{n=0}^{N-1} x[n]$$

$$X[0] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}0\, n} = \sum_{n=0}^{N-1} x[n]$$

$$X[N+\ell] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi(N+\ell)}{N}n} = \sum_{n=0}^{N-1} x[n]\, \boxed{e^{-j\frac{2\pi N}{N}n}}^{\,1}\, e^{-j\frac{2\pi \ell}{N}n}$$

$$= X[\ell] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi \ell}{N}n} \qquad \text{Q.E.D.}$$

* For real signals $X[k] = X^*[-k]$

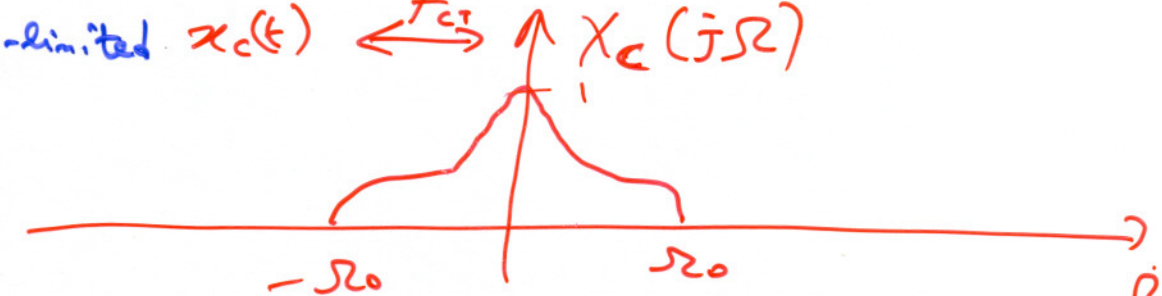
* $X[N-\ell] = X^*[\ell] \Rightarrow |X[N-\ell]| = |X[\ell]|$

$\measuredangle X[N-\ell] = \measuredangle X[\ell]$.

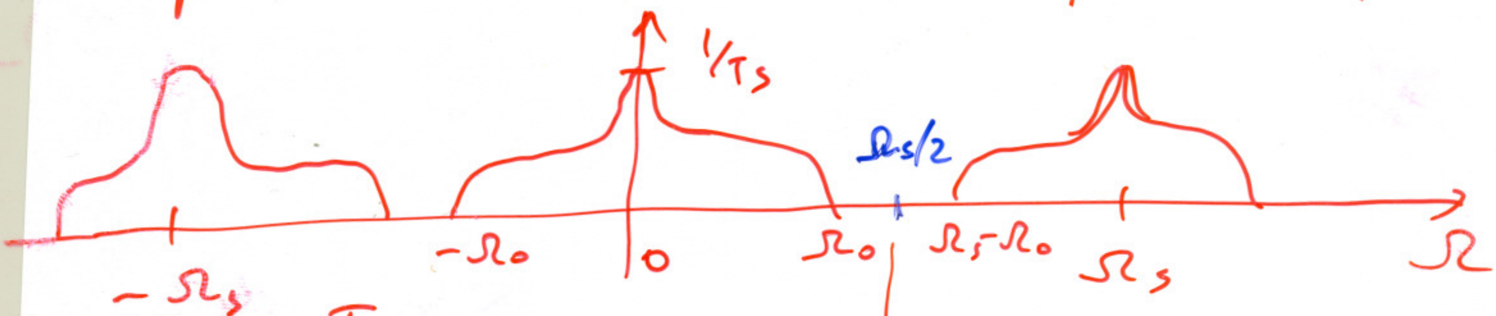
because $X(e^{j\omega})$ is 2π periodic!
and $X(e^{j\omega}) = X^*(e^{-j\omega})$.

Approximate Computation of ~~C~~ C.T.F.T (DFT–)

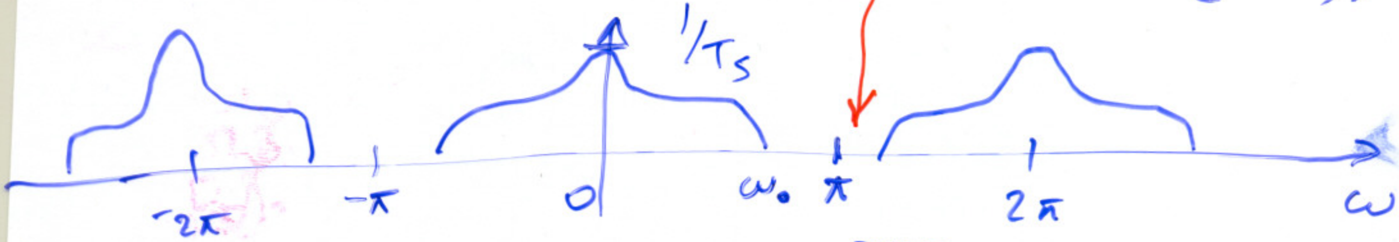
Band-limited $x_c(t) \xleftrightarrow{\mathcal{F}_{CT}} X_c(j\Omega)$



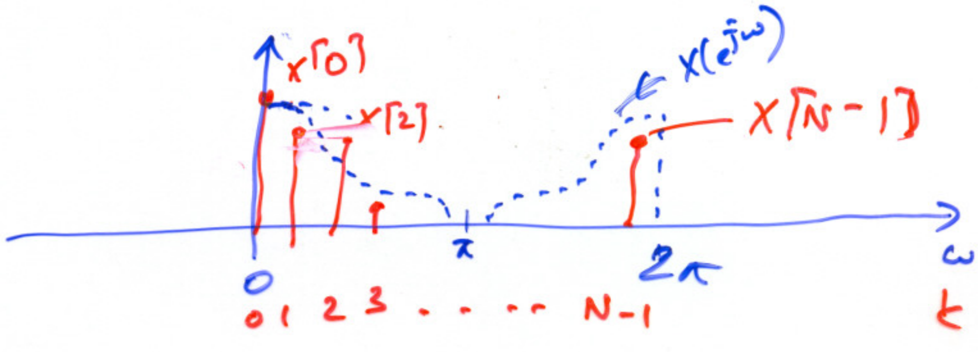
Sample this $\Omega_s > 2\Omega_0$: $x_p(t) \xleftrightarrow{\mathcal{F}} X_p(j\Omega)$



D.T.F.T.: $x[n] \xleftrightarrow{\mathcal{F}_{DT}} X(e^{j\omega})$ $\omega=\Omega T_s$ $x[n] = x_c(nT_s)$, $n=0, \mp 1, \mp 2$



We take $x[n]$, $n=0,1,\dots,N-1 \xleftrightarrow{DFT_N} X[k]$

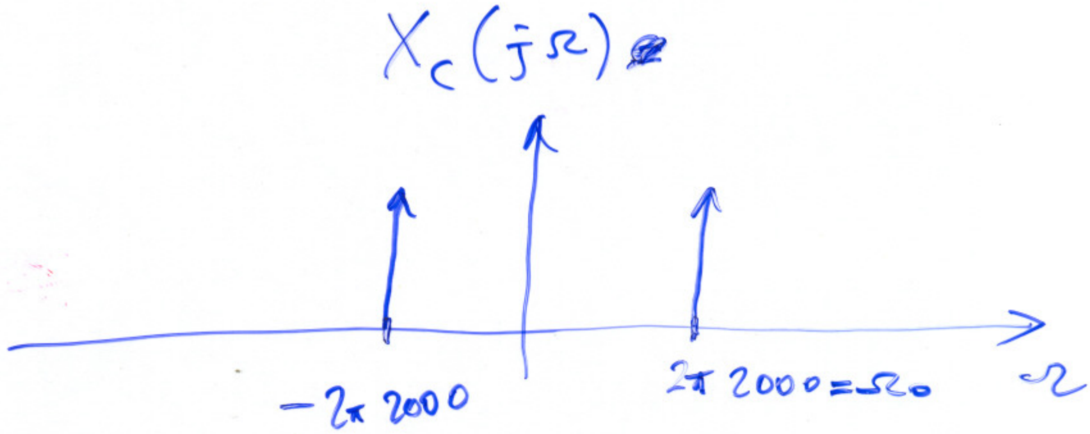


$$X[k] \cong X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$$

for $x[n] \approx 0$ for $n \geq N$ and $n<0$.

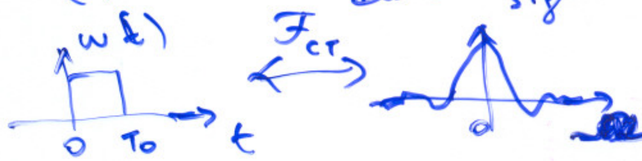
* Since $X(e^{j\omega})$ samples are related with C.T.F.T samples you can approximately compute C.T.F.T. samples as well!

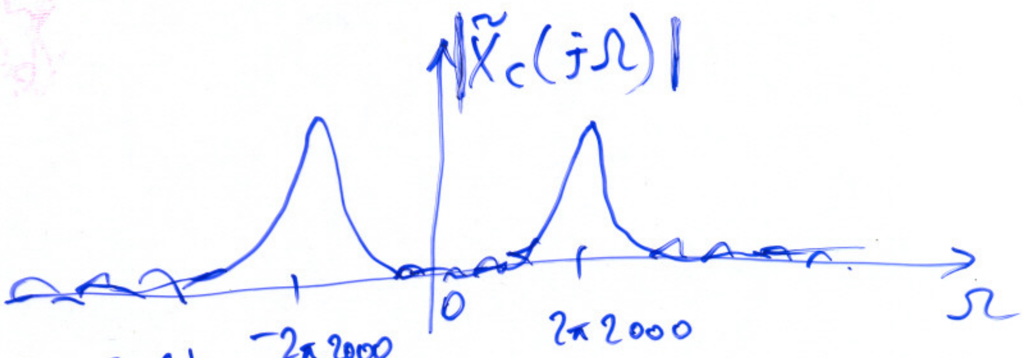
Ex|| $x_c(t) = \cos 2\pi\, 2000\, t$, sampling freq. $f_s = 8\,KHz$. $-\infty < t < \infty$.



In practice $\tilde{x}_c(t) = x_c(t)\, w(t)$ ← finite duration signal

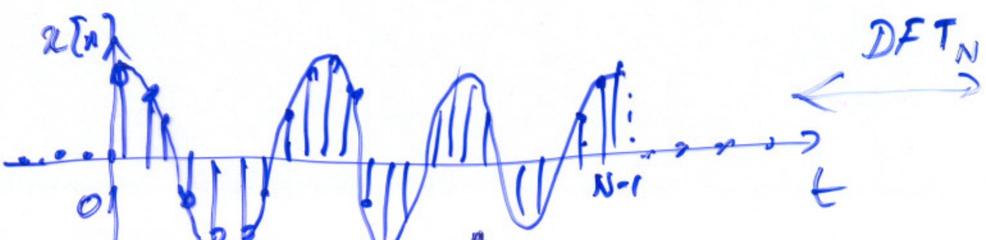
$$w(t) = \begin{cases} 1 & 0 < t < T_0 \\ 0 & \text{otherwise} \end{cases}$$





Sample $\tilde{x}_c(t)$

$x[n] = \tilde{x}_c(nT_s)$, $n = 0, 1, \dots, N-1$. $(NT_s \approx T_0)$

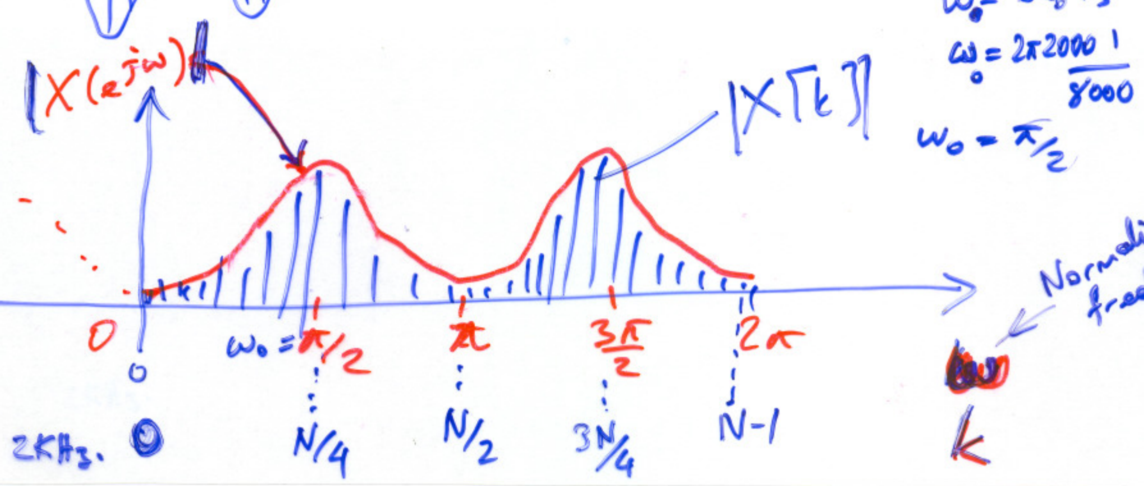


$\omega_0 = \Omega_0 T_s$

$\omega_0 = 2\pi 2000 \frac{1}{8000}$

$\omega_0 = \pi/2$

From the location of the peak $\frac{N}{4}$ in DFT domain we can determine the actual freq. 2KHz.



(DFT-5)

0	$2\pi 2000$	$2\pi 4000$	$-2\pi 2000$	$-2\pi 4000$	Ω
0	$\pi/2$	π	$3\pi/2$	2π	ω
0	$\frac{N}{4}$	$N/2$	$\frac{3N}{4}$	$N \ldots$	k

* From the location of the peak in DFT plot we can determine the frequency of the sinusoid $x_c(t) = \cos 2\pi 2000 t$

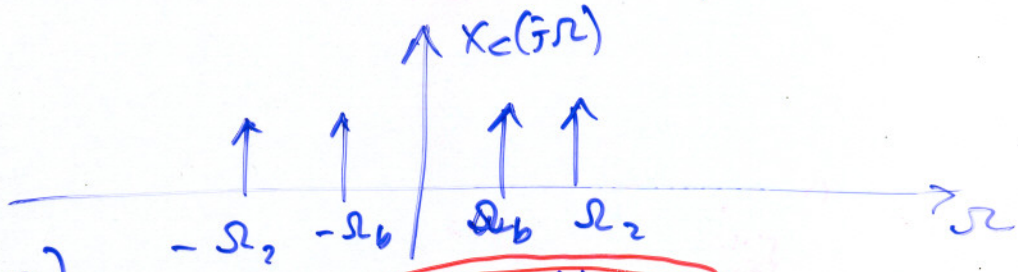
* (Mini-project) D.T.M.F = Dual-tone multi freq. (Dial)

	$\cos\Omega_1 t$	$\cos\Omega_2 t$	$\cos\Omega_3 t$	
$\cos\Omega_a t$	1	2	3	
$\cos\Omega_b t$	4	5	6	
$\cos\Omega_c t$	7	8	9	
$\cos\Omega_d t$	*	0	#	

Signals & Systems
McClellan

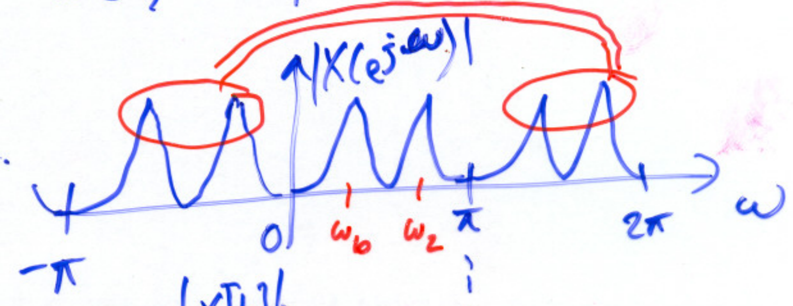
$0 < \Omega_1, \Omega_2, \Omega_3 < 2\pi 4\ KHz$
$\Omega_a, \Omega_b, \Omega_c, \Omega_d$

$$\text{"5"} = \cos\Omega_b t + \cos\Omega_2 t = x_c(t), \quad 0 < t \le T_0$$

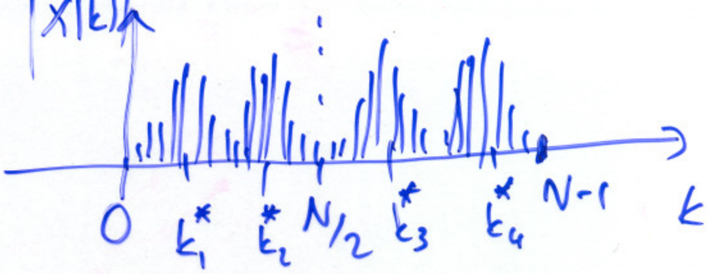


$T_s = \frac{1}{8000}$ sec

$x[n] = x_c(nT_s)$
$n = 0, 1, \ldots, N-1$



$\omega_b = \Omega_b \cdot T_s$
$\omega_2 = \Omega_2 T_s$



From k_1^* & k_2^* estimate Ω_b & Ω_2 $\Rightarrow$ Determine the number dialed!