Convolution using D.F.T.:

$N \geq M, L$ DFT size

$x_1[n]$, $n = 0, 1, \ldots, M-1$

$x_2[n]$, $n = 0, \ldots, L-1$

$x[n] = x_1[n] * x_2[n]$ $n = 0, 1, \ldots N-1$; Convolution length $M + L - 1 = N$

$$X[k] = X_1[k] \cdot X_2[k], \quad k = 0, 1, \ldots, N-1$$

Proof:

$$x_3[n] = \sum_{l=0}^{N-1} x_1[l]\, x_2[(n-l)]_N = x_1[n] \text{ Ⓝ } x_2[n] = x_3[n]$$

convolution

DFT

$$X_3[k] = \sum_{n=0}^{N-1} \underbrace{\left(\sum_{l=0}^{N-1} x_1[l]\, x_2[(n-l)]_L \right)}_{x_3[n]} e^{-j 2\pi \frac{kn}{N}}, \quad k = 0, 1, \ldots, L-1$$

$$X_3[k] = \sum_{l=0}^{N-1} x_1[l] \sum_{n=0}^{N-1} x_2[(n-l)]_L \, e^{-j\frac{2\pi k n}{N}}, \quad k=0,1,\dots$$

$m = n - l$

$$X_3[k] = \sum_{l=0}^{N-1} x_1[l] \sum_{m=0}^{L-1} x_2[m] \, e^{-j\frac{2\pi k (m+l)}{N}}$$

$x_2[(m)_L] = x_2[m]$

$$X_3[k] = \sum_{l} x_1[l] \left(\sum_{m} x_2[m] \, e^{-j\frac{2\pi k}{N} m} \right) e^{-j\frac{2\pi k}{N} l}$$

$$X_3[k] = \underbrace{\sum_{l} x_1(l) \, e^{-j\frac{2\pi k}{N} l}}_{} \; \underbrace{\sum_{m} x_2[m] \, e^{-j\frac{2\pi k m}{N}}}_{X_2[k]}$$

$$X_3[k] = X_1[k] \cdot X_2[k]$$

$x_1[n] \xleftrightarrow{L-DFT} \bar{X}_1[k]$ $\qquad L \geq M$

$x_2[n] \xleftrightarrow{L-DFT} \bar{X}_2[k]$ $\qquad N \geq L$

$N = L + M - 1$

$$x_3[n] = x_1[n] \; (L) \; x_2[n]$$

$$\bar{X}_3[k] = \bar{X}_1[k] \cdot \bar{X}_2[k]$$

$k = 0, 1, \dots, L-1.$

Circular Convolution

$$x_3[m] = \sum_{n=0}^{L-1} x_1[n] \, x_2[(m-n)_L]$$

modulo L

Regular Conv.

$$y[m] = \sum_{n=\cancel{-\infty}\,0}^{\cancel{\infty}\,L-1} x_1[n] \, x_2[m-n]$$

if $x_1[n] = 0$ for $n < 0$ & $n \geq L$

Circular Convolution :

Ex

$$x_{\circ}[n] = \begin{cases} 1, & n=0,1 \\ 0, & \text{o.w.} \end{cases}$$

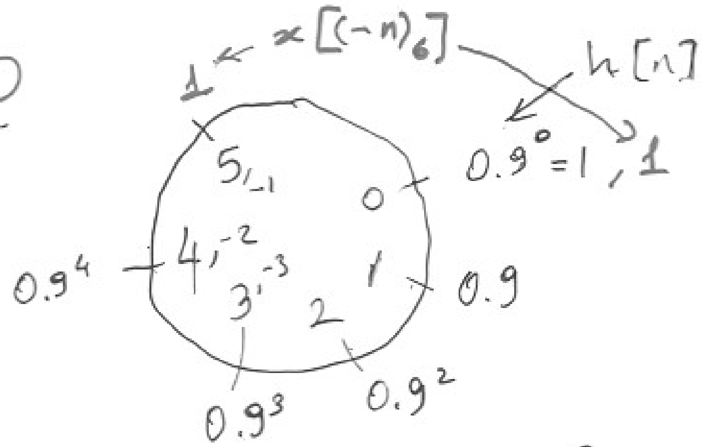
$$h[n] = \begin{cases} 0.9^n, & n=0,1,\dots,4 \\ 0 & \text{o.w.} \end{cases}$$

$\Rightarrow x_3[m] = \sum h[n] \, x[(m-n)$

$x_3[n] = x[n] \circledast_6 h[n]$.

($y[n] = x[n] * h[n]$ has a length of 6.)

$n=0$



$$x_3[0] = \sum_{n=0}^{N-1=5} h[n] \, x[(0-n)_6]$$

$x[(0)_6] = x[0]$

$x[(-1)_6] = x[(5)_6] = 0$

$x[(-2)_6] = 0$

$\vdots$

$$x_3[0] = 0.9^0 = y[0] = 1$$

$n=1$

$$x_3[1] = \sum_{n=0}^{5} h[n] \, x[((1-n)_6] = h[0] x[(1)_6] + h[1] x[(0)_6]$$

$$x_3[1] = h[0] \, x[1] + h[1] \, x[0] = y[1] = 1 + 0.9$$

$n=2$

$$x_3[2] = \sum_{n=0}^{5} h[n] \, x[(2-n)_6] = h[0] \underbrace{x[2]}_{=0} + h[1] x[1] + h[2] x[0] + h[3] \underbrace{x[(-1)_6]}_{=0}$$

$$= 0.9 + 0.9^2 = y[2]$$

$n=3$ $\quad x_3[3] = 0.9^3 + 0.9^2 = y[3]$

$n=4$ $\quad x_3[4] = 0.9^4 + 0.9^3 = y[4]$

$n=5$ $\quad y[5] = x_3[5] = \sum_{n=0}^{5} h[n] \, x[(5-n)_6] = h[0] \, \underbrace{x[(5)_6]}_{=0}$

$+ h[1] x[(4)_6] = 0$

$+ h[2] x[(3)_6] = 0$

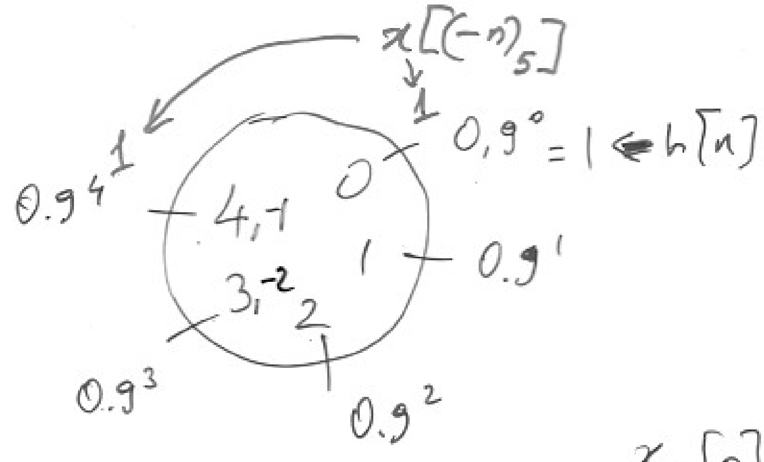
$+ h[3] x[(2)_6] = 0$

$+ h[4] x[(1)_6] = 0.9^4$

$n=6$ $\quad x_3[(6)_6] = x_3[0]$:

$\Rightarrow x_3[n] = y[n]$ for $n=0,1,\dots,5$

For $M=5$ we have a problem.



$$x_2[n] = x[n] \, ⑤ \, h[n]$$

$$x_2[0] = 1 + 0.9^4$$

$$x_2[0] = \sum_{n=0}^{4} h[n]\, x[(-n)_{\mathrm{mod}\,5}] = h[0]\,x[0]^{=1}$$

$$+ h[1]\,x[(-1)_5]^{=0}$$

$$+ h[2]\,x[(-2)_5] + h[3]\,x[(-3)_5]$$

$$+ h[4]\,x[(-4)_5]$$

$$= 1 + 0.9^4$$

$(0 = x[(-1)_5] = x[4]$

$(x[1] = x[(-4)_5] = 1)$

$x_2[1] = y[1]$, $x_2[2] = y[2]$; $x_2[3] = y[3]$

$x_2[4] = y[4]$ $x_2[5] = x_2[0] \Rightarrow$ useless

$$x_2[0] = y[0] + y[5]$$

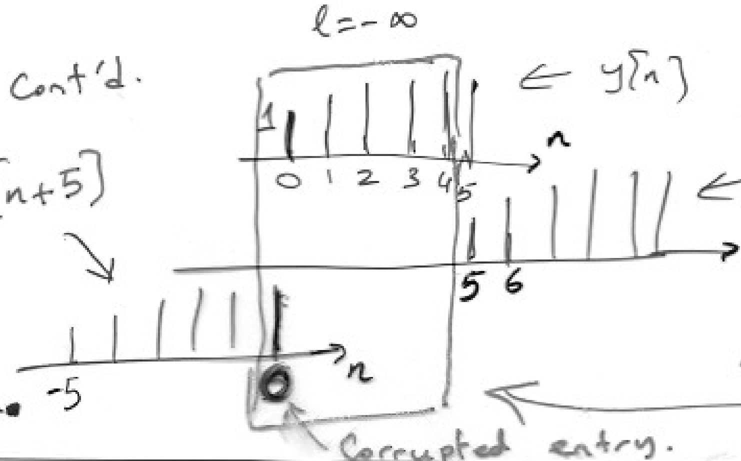
$\leftarrow$ corrupting term.

Let $y[n] = x_1[n] * x_2[n]$, $x_3[n] = x_1[n] \, Ⓜ \, x_2[n]$

periodic with period M

$$x_3[n] = \sum_{l=-\infty}^{\infty} y[n - lM] = y[n] + y[n-M] + \cdots + y[n+M] + \cdots$$

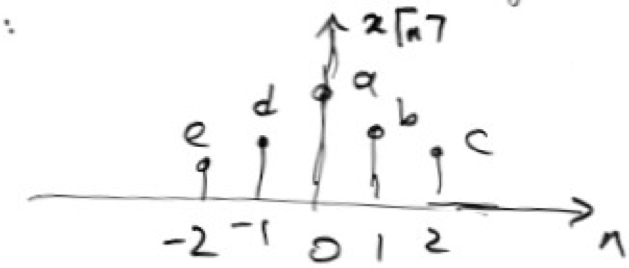
Ex11 cont'd.



$M=5$

x_3 has a period of 5

Ex|| Compute the DFT of a two-sided signal:

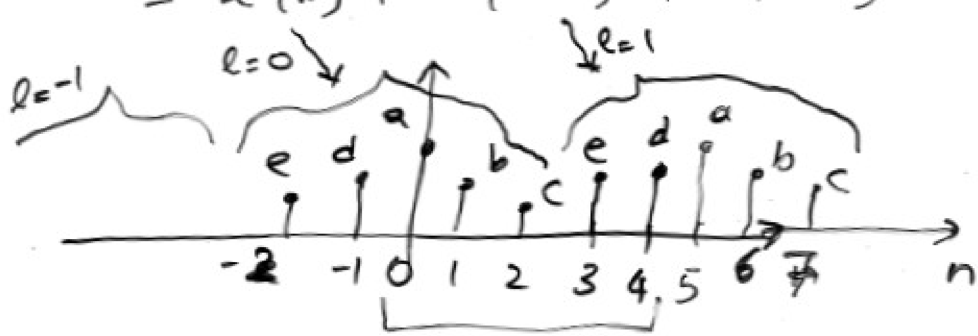


1) Shift this signal $\bar{x}[n] = x[n-2]$ and compute the DFT of $\bar{x}[n]$ & continue

2^{nd} approach) Assume a periodic extension

$$x_p[n] = \sum_{l=-\infty}^{\infty} x[n - lN] \qquad N \geq 5$$

$$= x[n] + x[n-5] + x[n+5] + \cdots$$



$x_p[n] = \{a, b, c, \underset{\substack{\uparrow \\ -2^{nd}}}{e}, \underset{\substack{\uparrow \\ -1^{st}}}{d}\}$ for $n = 0, 1, \ldots, 4$

$$\bar{x}[n] \xleftrightarrow{DFT_N} \bar{X}[k]$$

$$x_p[n] \xleftrightarrow{DFT_N} X_p[k]$$

Ex/ cont'd.

$$|X_p[k]| = |\bar{X}[k]|$$

Only linear phase term difference between the two. DFT's.

linear phase term

Property:

$$\text{length } L \le N \quad x[n] \xleftrightarrow{DFT_N} X[k]$$

$$x[(n-m)_N] \xleftrightarrow{DFT_N} X[k]\, e^{-j\frac{2\pi k m}{N}}$$

(Due to shift $x[n-m]$ may have non-zero term after the index N. Therefore we need $(n-m)_N$ to keep all the coefficients of $x[(n-m)_N]$ in the range of $n = 0, 1, \ldots, N-1$.)

Linearity property:

For all $\alpha, \beta \in \mathbb{R}$ and x_1 & x_2

$$x_1[n] \xleftrightarrow{DFT_N} X_1[k]$$

$$x_2[n] \xleftrightarrow{DFT_N} X_2[k]$$

$$\alpha\, x_1[n] + \beta\, x_2[n] \xleftrightarrow{DFT_N} \alpha\, X_1[k] + \beta\, X_2[k], \quad k = 0, 1, \ldots, N-$$

(DTFT:

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$x[n-m] \longleftrightarrow X(e^{j\omega})\, e^{-j\omega m}$$

)

Ex 11

$$x[n] = \begin{cases} 1, & n=m \\ 0, & o.w. \end{cases}$$

Compute N-point DFT of $x[n]$.

$$X[0] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{N-1} x[n] = 1 = x[\cdot] = 1$$

$$X[1] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}} = x[m] e^{-j\frac{2\pi 1 m}{N}} = e^{-j\frac{2\pi m}{N}} = W_N^{m}$$

$$X[2] = W_N^{2m} = \overset{=1}{x[m]} e^{-j\frac{2\pi 2 m}{N}}$$

$\vdots$

$$X[N-1] = W_N^{(N-1)m}$$

So: $X[k] = W_N^{km} = e^{-j\frac{2\pi}{N}km}$

$DTFT\{x[n]\} = X(e^{j\omega}) = e^{-j\omega m}$ for all

Inverse D.F.T.

$$x[n] = \begin{cases} 1 & \text{for } n=m \\ 0, & o.w. \end{cases} = \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{e^{-j\frac{2\pi}{N}km}}_{=X[k]} \cdot e^{j\frac{2\pi}{N}kn}$$

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)\cdot k} = \begin{cases} N & \text{for } n-m = 0, \mp N, \mp 2N \dots \\ 0. & o.w. \end{cases}$$

Ex 4

$$x[n] = \cos\frac{2\pi r n}{N}, \quad 0 \le n \le N-1$$

($r = 0, 1, 2, \dots, N-1$. can take any one of th

$$x[n] = \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

$$X[k] = \frac{1}{2} \underbrace{\sum_{n=0}^{N-1} W_N^{(r-k)n}}_{\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(r-k)n}} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{(r+k)n}$$

$$X[k] = \begin{cases} N/2, & k=r \\ N/2, & k=N-r \\ 0 & o.w. \end{cases}$$

* Circular Convolution:

$$X_1[k] = \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi nk}{N}} \quad k=0,1,\dots,N-1. \quad \text{and} \quad X_2[k] \xleftrightarrow{N} x_2[n]$$

$$X_3[k] = X_1[k]\, X_2[k] \quad , k=0,1,2,\dots,N-1$$

IDFT $\Updownarrow N$

$$x_3[m] = \sum_{n=0}^{N-1} x_1[n]\, x_2[(m-n)_N] \quad m=0,1,\dots,N-1.$$

→ No ordinary convolution.

or $x_{p2}[n] = \sum_{l=\infty}^{\infty} x_2(m-lN)$ and convolve $x_3 * x_{p2}[m]$, $m=0,1,\dots,N-$

$(m-n)_N$ = modulo $(m-n)_N$

Ex

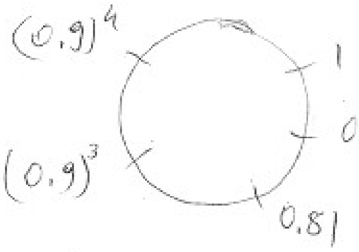
$x_2[(N)_N] = x_2[0] \neq 0$.

($x_2[N] = 0$.)

$x_2[(N+1)_N] = x_2[1]$ etc.

$$x[n] = \begin{cases} 1 & n=0,1 \\ 0 & \text{o.w.} \end{cases} \quad , \quad h[n] = \begin{cases} 0.9^n & , 0,1,\dots,4. \\ 0 & \end{cases}$$

$$y[n] = x[n] \; ⑤ \; h[n]$$



$y[0] = 1+(0.9)^4$ → corrupted

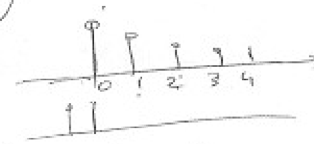
$y[1] = 1+0.9$

$y[2] = 0.9+0.9^2$

$y[3] = (0.9)^2+(0.9)^3$

$y[4] = (0.9)^4+0.9^3$ and ($y[0]$ is corrupted. No $y[5]$)

Regular convolution: $z[n] = x[n] * h[n]$.



$z(0)=1$, $z[1] = 1+0.9$, $z[2] = 0.9+0.9^2$, $z[3] = 0.9^2+0.9^3$

$z[4] = 0.9^4+0.9^3$

$z(5) = 0.9^4$

* length $[h[n] * u[n]] = L_1 + L_2 - 1$ $\quad$ (ex. $7+2-1 = 6$)

* DFT size $\geq L_1 + L_2 - 1 = L$ $\quad$ $\cancel{x_3[m] = x_3[m]}$.

$x_1(n) \; Ⓛ \; x_2(n) = x_1(n) * x_2(n)$.

DFT: $x[n] \xleftrightarrow[N]{DFT} X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$ | Oct 25 | 1

$x[n]$ is a finite extent signal whose duration is less than N.

$X[k] = X(e^{j\omega})|_{\omega = w_N^k}$

* $x[n]$ is an arbitrary signal: Sampling in the Fourier Domain:

$$X[k] = X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{2\pi kn}{N}}$$

$$= \cdots + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi kn}{N}} + \sum_{n=-N}^{-1} x[n] e^{-j\frac{2\pi kn}{N}} + \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n} + \cdots$$

or

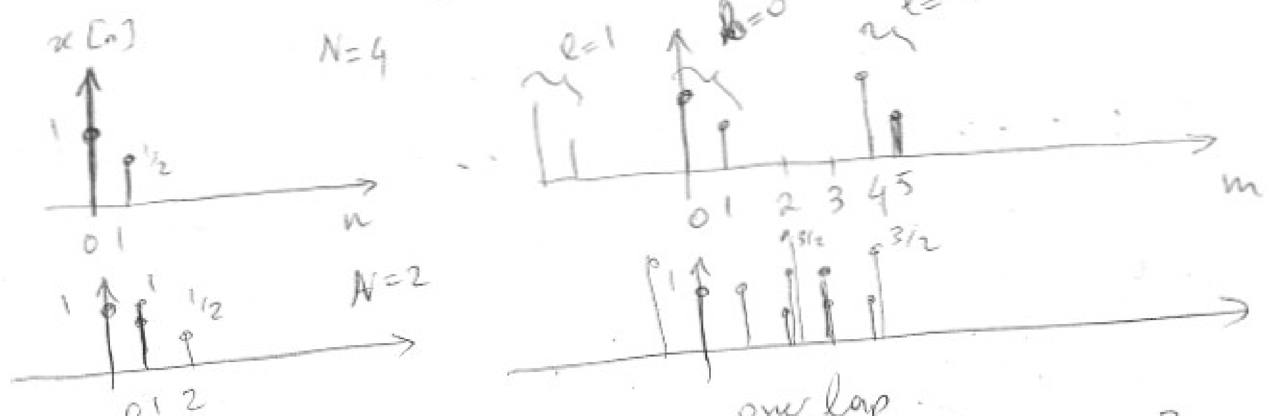
$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n] e^{-j\frac{2\pi kn}{N}} = \sum_{l} \sum_{m=0}^{N-1} x[m+lN] e^{-j\frac{2\pi km}{N}}$$

Let $m = n - lN$

$$X[k] = \sum_{m=0}^{N-1} \left(\sum_{l=-\infty}^{\infty} x[m+lN] \right) e^{-j\frac{2\pi km}{N}}$$

The signal $x_p[m] = \sum_{l=-\infty}^{\infty} x[m \mp lN]$ is periodic with period N.

Ex//



$$x_p[m] = \sum_{l=-\infty}^{\infty} x[m - lN] = \sum_{l} x[m + lM].$$

Study Example 5.1.1.

* Linearity: $a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow[N]{DFT} a_1 X_1[k] + a_2 X_2[k]$.

Properties: * $X[k] = X^*[N-k]$ for real $x[n]$.

$X(e^{j\omega}) = X^*(e^{-j\omega})$ " "

$X(e^{j\omega}) = X^*(e^{j(2\pi-\omega)})$