Inverse D.F.T.

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \underline{W}_N^{-1} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\underline{W}_N^{-1} = \frac{1}{N} \underline{W}_N^{*}$$ ← complex conjugate transpose

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j \frac{2\pi k}{N} n}, \quad n = 0, 1, \ldots, N-1.$$

(Inv. D.T.F.T: $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$)

* What happens if n is outside the index set $0, 1, \ldots N-1$

Define: $x[N] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j \frac{2\pi k}{\cancel{N}} \cancel{N}} = \frac{1}{N} \sum_{k=0}^{N-1} X[k]$

$x[0] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j \frac{2\pi}{N} k \cdot 0} = \frac{1}{N} \sum_{k=0}^{N-1} X[k]$

Define $x[N+1] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} (N+1)} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N}} = x[1]$

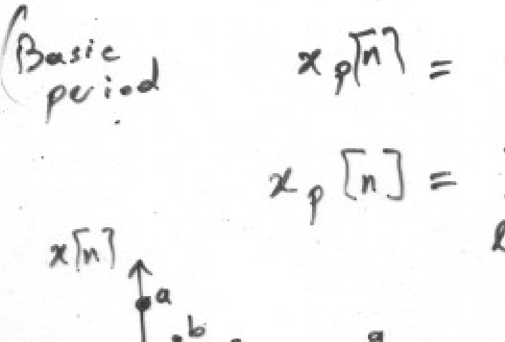
Define $x[-1] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j \frac{2\pi k}{N} (-1)} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{+j \frac{2\pi}{N} (N-1)} = x[N-1]$

$\Rightarrow$ periodic extension: $x_p[n]$

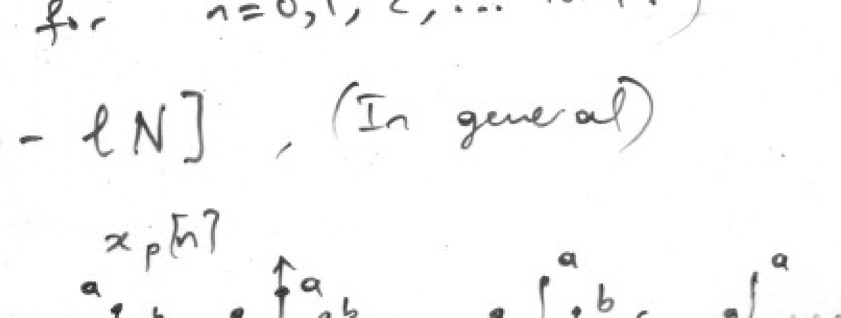
$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j \frac{2\pi k}{N} n}, \quad n = 0, \mp 1, \mp 2, \ldots$$

(Basic period $x_p[n] = x[n]$, for $n = 0, 1, 2, \ldots N-1$.)

$$x_p[n] = \sum_{\ell=-\infty}^{\infty} x[n - \ell N] \quad \text{(In general)}$$



$\longrightarrow$



FFT for $N = 3p$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \qquad k = 0, 1, 2, \dots, N-1$$

Decimation-in-time:

$$X[k] = \sum_{l=0}^{N/3-1} x[3l]\, e^{-j\frac{2\pi k}{N}3l} + \sum_{l=0}^{\frac{N}{3}-1} x[3l+1]\, e^{-j\frac{2\pi k}{N}(3l+1)} + \sum_{l=0}^{\frac{N}{3}-1} x[3l+2]\, e^{-j\frac{2\pi k}{N}(3l+2)}$$

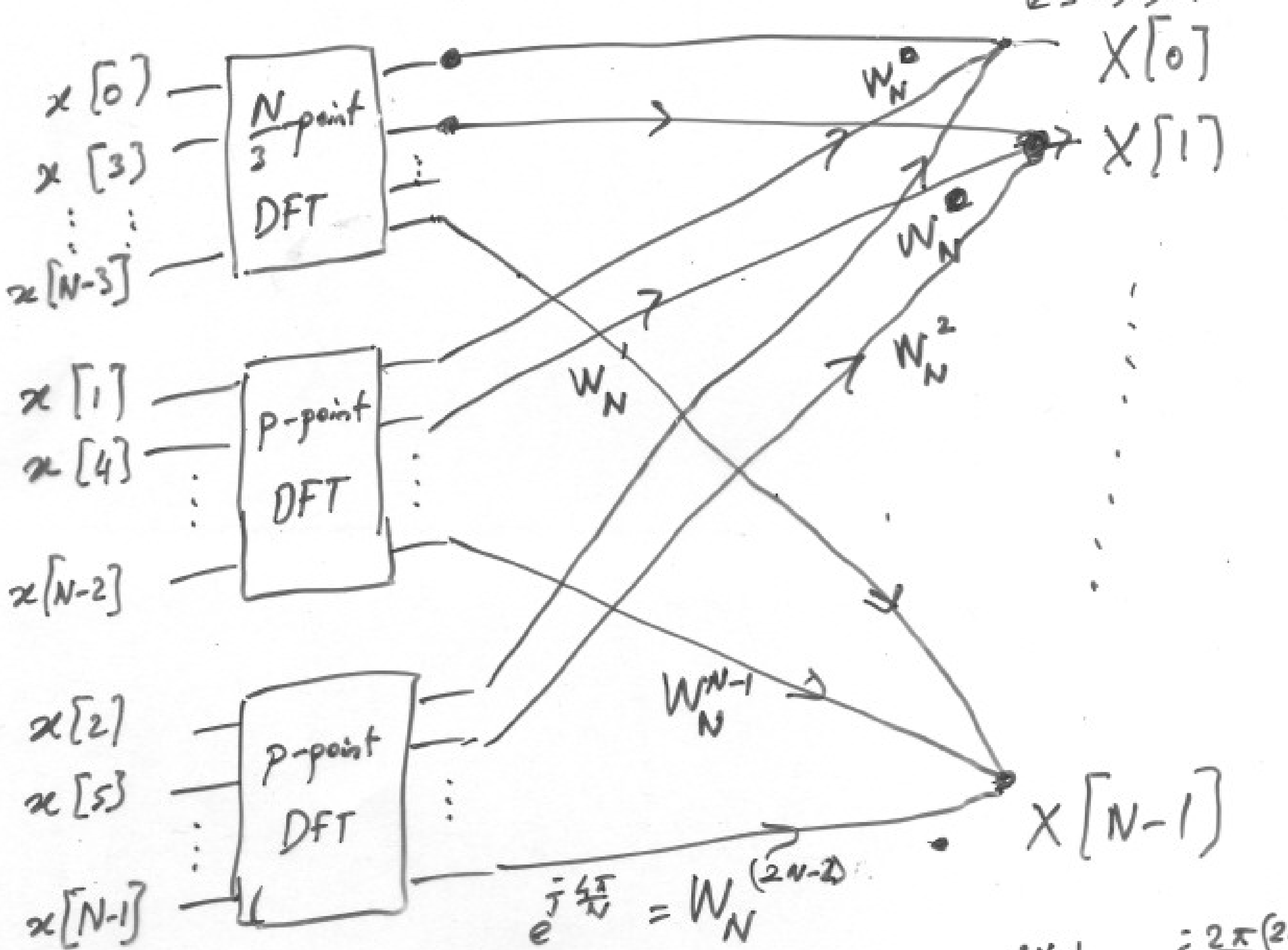
$(x[0], x[3], \dots)$ $(x[1], x[4], \dots)$ $(x[2], x[5], \dots)$

$$X[k] = \sum_{l=0}^{\frac{N}{3}-1} x[3l]\, e^{-j\frac{2\pi k}{(N/3)}l} + e^{-j\frac{2\pi k}{N}} \sum_{l=0}^{\frac{N}{3}-1} x[3l+1]\, e^{-j\frac{2\pi k}{(N/3)}l} + e^{-j\frac{2\pi k}{N}2} \sum_{l=0}^{\frac{N}{3}-1} x[3l+2]\, e^{-j\frac{2\pi k}{(N/3)}l}$$

$\frac{N}{3}$ - DFT

$\frac{N}{3} = p$ - point, p-point DFT's.

$k = 0, 1, 2, \dots, N-1$



$$W_N^{2N-1} = e^{-j\frac{2\pi(2N-2)}{N}} = e^{+j\frac{4\pi}{N}}$$

Inverse D.F.T.: $x(n) = \frac{1}{N} \sum_{k=0}^{N} X[k] W_N^{-kn}$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi k n}{N}} \quad , \; n=0,1,\dots,N-1$$

Proof:

Forward D.F.T:

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & \cdots & e^{-j\frac{2\pi(N-1)}{N}} \\ \vdots & \vdots & & \\ 1 & e^{-j\frac{2\pi(N-1)}{N}} & \cdots & e^{-j\frac{2\pi(N-1)^2}{N}} \end{bmatrix}}_{= \underline{W}_N} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

(The last row $[1 \;\; e^{-j2\pi\frac{(N-1)}{N}} \;\; e^{-j\frac{2\pi(N-2)}{N}} \; \cdots \; e^{-j\frac{2\pi}{N}}]$)

* $\underline{W}_N$ is a symmetric matrix.
* Rows ~~of~~ $\underline{W}_N$ are orthogonal to each oth

$$W_N^{-1} = \frac{1}{N} W_N^H = \frac{1}{N}\left(W_N^*\right)^T = \frac{1}{N} W_N^*$$

$$\Longrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi k n}{N}} \quad n=0,1,\dots,N-1$$

Ex//

$$x[n] = \begin{cases} 1, & n=0 \\ 0, & \text{o.w.} \end{cases} \longrightarrow X[k] = ? \quad , \; k=0,\dots,N-$$

$$X[0] = \sum_{n=0}^{N-1} x[n] = 1$$

$$X[1] = \sum_n x[n]\, e^{-j\frac{2\pi 1 n}{N}} = 1 \cdot e^{-j\frac{2\pi 1 \cdot 0}{N}} + 0 + \cdots + 0 = $$

$$\vdots$$

$$X[N-1] = \sum_n x[n]\, e^{-j\frac{2\pi (N-1)\cdot n}{N}} = 1\, e^{-j\frac{2\pi(N-1)0}{N}} + 0 + \cdots + 0 = 1$$

Inverse D.F.T. expression

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{1}_{= X[k]}\, e^{+j\frac{2\pi \cdot kn}{N}} = \begin{cases} 1 & \text{for } n = 0, \mp N, 2N, \mp 3N \\ 0 & \text{o.w. } (n=1,2,\dots,N-1) \end{cases}$$

FFT 5

Fast Fourier Transform (FFT) Algorithm

DFT: $$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}n}, \quad k = 0, 1, \ldots, N-1$$

Direct implementation comp. cost : N^2 (complex multiplications) + N additions

* If $N = 1024 = 2^{10} \Rightarrow N^2 \cong 10^3 \times 10^3 = 10^6$ multiplications.
* FFT $O(N \log_2 N)$ complex multiplications $\cong \frac{10^3}{2} \log_2 1024 \cong \frac{10^4}{2}$ real multipl

Decimation-in-time FFT algorithm: $\left(W_N^k = e^{-j\frac{2\pi k}{N}}\right)$
(N is a power of 2)

$$X[k] = \sum_{\substack{n=0 \\ n,\ \text{even}}}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}n} + \sum_{\substack{n=0 \\ n,\ \text{odd}}}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}}, \quad k = 0, 1, \ldots, N-1$$

$$X[k] = \sum_{l=0}^{N/2-1} x[2l]\, e^{-j\frac{2\pi k}{N}2l} + \sum_{l=0}^{N/2-1} x[2l+1]\, e^{-j\frac{2\pi k}{N}(2l+1)}$$

$$X[k] = \sum_{l=0}^{\frac{N}{2}-1} x[2l] \left(W_N^2\right)^{lk} + W_N^k \sum_{l=0}^{\frac{N}{2}-1} x[2l+1]\, \underbrace{e^{-j\frac{2\pi k}{N}2l}}_{=\left(W_N^2\right)^{kl}}$$

$$W_N^2 = W_{N/2} = e^{-j\frac{2\pi k 2}{N}} = e^{-j\frac{2\pi k}{N/2}} = W_{N/2}$$

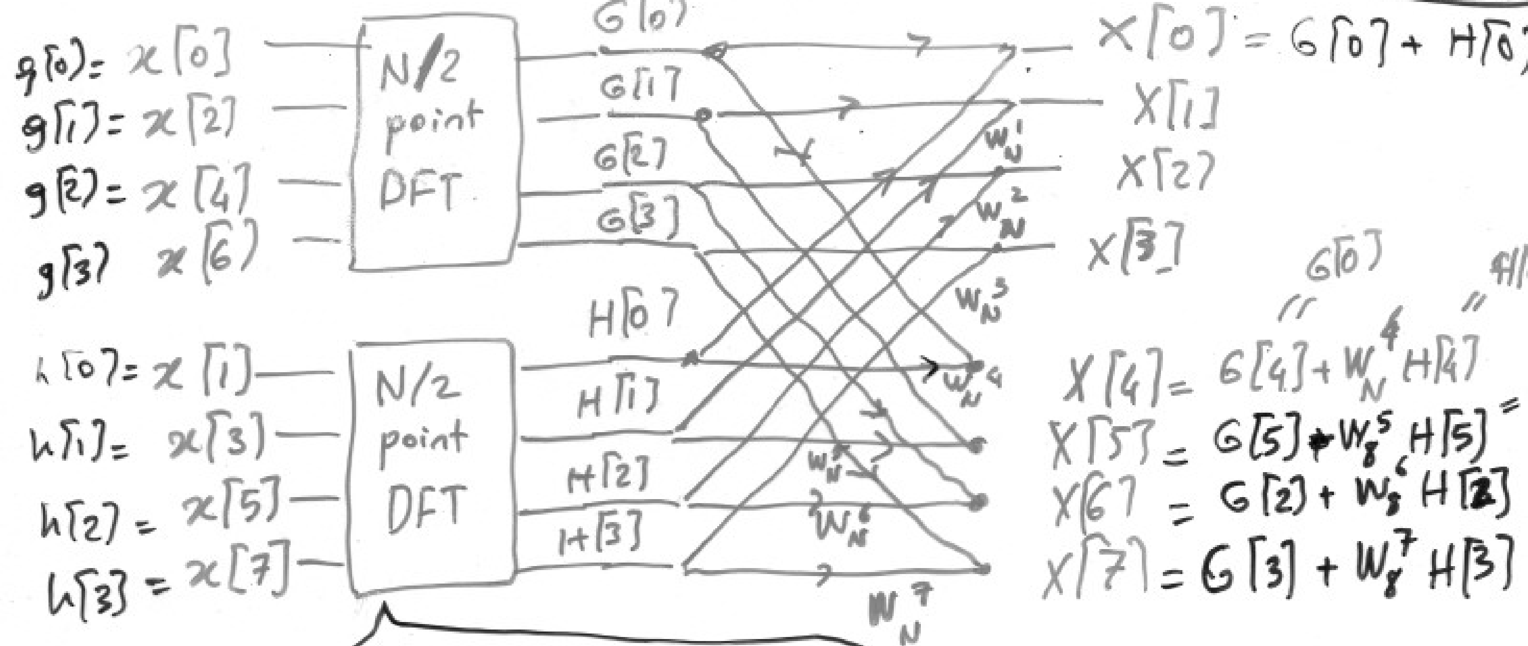
$$X[k] = \underbrace{\sum_{l=0}^{N/2-1} x[2l]\, W_{N/2}^{lk}}_{N/2 \text{ point DFT of evens}} + W_N^k \underbrace{\sum_{l=0}^{N/2-1} x[2l+1]\, W_{N/2}^{lk}}_{N/2 \text{ point DFT of odds}}$$

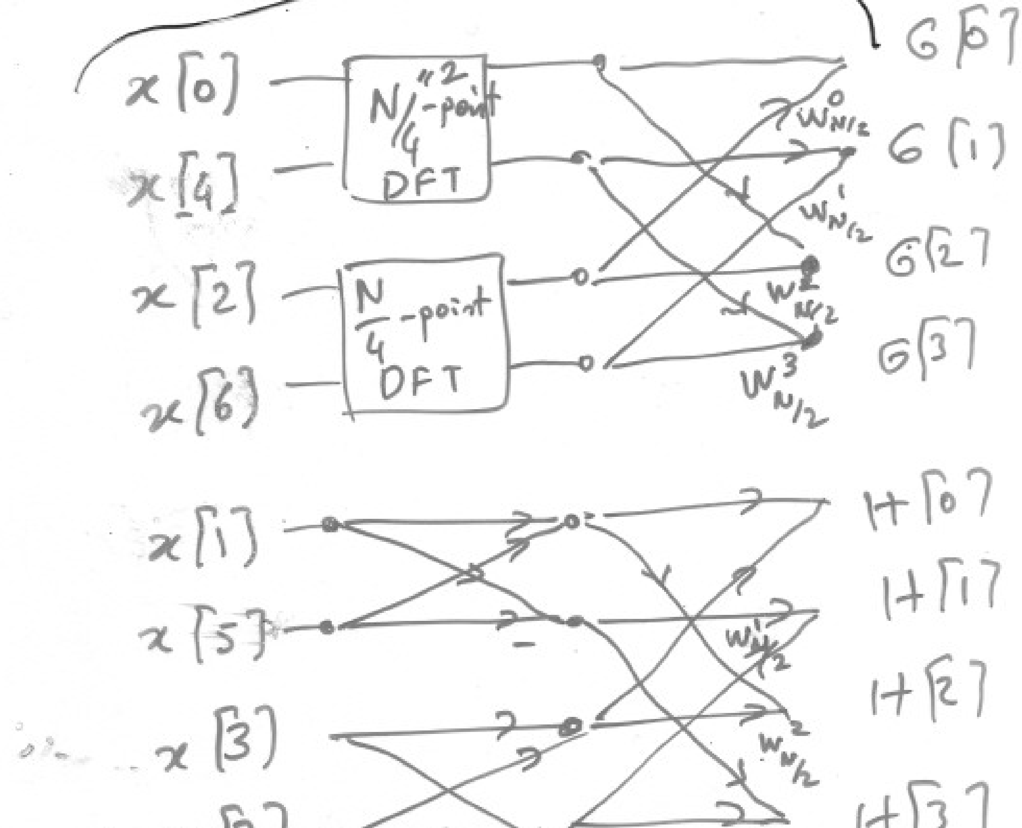
$$X[k] = G[k] + W_N^k H[k], \quad k = 0, 1, \ldots, N-1 \qquad (1*)$$

Comp. cost study:
($N = 1024 = 2^{10} \cong 10^3 \Rightarrow X[k]$: $\underbrace{(512)^2}_{G[k]} + \underbrace{(512)^2}_{H[k]} + 10^3$ $\qquad \underbrace{10^6}_{X[k]}$

periodic with N/2 000 $< 10^6$

N=8





Decimation-in-time 8-point DFT flowchart.

* $\log_2 N$ stages. $(2^3 = 8 \Rightarrow 3$ stages$)$

* Input : bit reversed order $\Rightarrow$ output: regular

$x[6]$ - - reverse bits $X[3]$

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