General Transformation Topic

Basis functions

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot \phi_k^*[n] \quad \& \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \phi_k[n]$$

Orthogonal Basis $\Rightarrow \frac{1}{N} \sum_{n=0}^{N-1} \phi_k[n] \cdot \phi_m^*[n] = \begin{cases} 1 & k=m \\ 0 & k \neq m \end{cases}$

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* In case of DCT we have cosine basis.

Cosines are 1) Real Functions

$\boxed{\text{2) Even Symmetric} \quad \text{3) Periodic}}$

$\boxed{\text{DFT} \Rightarrow \text{Periodicity of the transformed signal is assumed.}}$

• In DFT, what we do is to form a periodic sequence from the finite length signal

• In DCT $\Rightarrow$ We form a even symmetric and Periodic sequence from the finite length Signal.

Example

$x[n] = \{a, b, c, d\}$
$\uparrow$
$n=0$

1) $\tilde{x}_1[n] = \{a, b, c, d, c, b, a, b, c, d, c, b, a\}$

2) $\tilde{x}_2[n] = \{a, b, c, d, d, c, b, a, a, b, c, d, d, c, b, a, a\}$

3) $\tilde{x}_3[n] = \{a, b, c, d, 0, -d, -c, -b, -a, -b, -c, -d, 0, d, c, b, a\}$

4) $\tilde{x}_4[n] = \{a, b, c, d, 0, -d, -c, -b, -a, -a, -b, -c, -d, 0, d, c, b, a, a\}$

All of these signals are periodic with N=16 or less. and they have even symmetry.

* The first step in DCT is to form one of these periodic, even symmetric sequences. Therefore we have 4 different definitions of DCT. Most commonly used ones are DCT-1 and DCT-2 which includes $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$

$$\tilde{x}_1[n] = x_\alpha[(n)_{2N-2}] + x_\alpha[(-n)_{2N-2}] \quad \text{where} \quad x_\alpha[n] = \alpha[n] \cdot x[n]$$

$$\alpha[n] = \begin{cases} 1/2 & n=0,\ n=N-1 \\ 1 & \text{otw.} \end{cases}$$

Weighting of the endpoints

Because doubling occurs at the end points.

DCT-1 is defined as

$$X^{c1}[k] = 2\sum_{n=0}^{N-1} \alpha[n] \cdot x[n] \cdot \cos\left(\frac{\pi k n}{N-1}\right) \quad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N-1} \sum_{k=0}^{N-1} \alpha[k]\, X^{c1}[k] \cdot \cos\left(\frac{\pi k n}{N-1}\right) \quad 0 \le k \le N-1$$

where $\alpha[n] = \begin{cases} 1/2 & n=0, N-1 \\ 1 & \text{o.w.} \end{cases}$

$\tilde{x}_2[n] = x[(n)_{2N}] + x[(-n-1)_{2N}] \Rightarrow$ No modifications since the end points do not overlap.

$$X^{c2}[k] = 2\sum_{n=0}^{N-1} x[n] \cdot \cos\left(\frac{\pi k(2n+1)}{2N}\right) \quad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} \beta[k] \cdot X^{c2}[k] \cdot \cos\left(\frac{\pi k(2n+1)}{2N}\right)$$

$$\beta[k] = \begin{cases} \frac{1}{2} & k=0 \\ 1 & 1 \le k \le N-1 \end{cases}$$

Relationship between DFT and DCT

Obviously for different definitions of DCT (as DCT-1 and DCT-2), there exist different relationships.

* Relation between DFT & DCT-1

a) Time Domain → Transform Domain

From the former sections we know that

$$\tilde{x}_1[n] = x_\alpha[(n)_{2N-2}] + x_\alpha[(-n)_{2N-2}] \quad , n = 0, 1 \dots 2N-3$$

where $x_\alpha[n] = \alpha[n] . x[n]$

- Assume that $X_\alpha[k]$ is the $\underline{(2N-2)}$ point DFT of $x_\alpha[n]$

$$\Rightarrow X_1[k] = X_\alpha[k] + X_\alpha^*[k] = 2\,\mathrm{Re}\{X_\alpha[k]\} \quad k = 0, 1 \dots 2N-3$$

2N-2 point DFT of $x_1[n]$

$$= 2 . \sum_{n=0}^{N-1} \alpha[n] . x[n] . \cos\left(\frac{2\pi k n}{2N-2}\right) = X^{c1}[k]$$

example: $x[n] = \{a, b, c, d\} \Rightarrow \tilde{x}_1[n] = \{a/2, b, c, \frac{d}{2}, 0, 0\} + \{a/2, 0, 0, \frac{d}{2}, c, b\}$

$$= \{a, b, c, d, c, b\}$$

$$\Rightarrow X_1[k] = \sum_{n=0}^{5} \tilde{x}_1[n] . e^{\frac{-j2\pi k n}{6}} = a + b.e^{-j2\pi k/6} + c e^{-j4\pi k/6} + d(-1)^k + b.e^{-j10\pi k/6} + c e^{-j8\pi k/6}$$

cos cos

Conclusion: DCT-1 of an N point sequence is identical to the $(2N-2)$ point DFT of the symmetrically extended sequence $x_1[n]$ and it is also identical to twice the real part of the first N points of the $(2N-2)$-point DFT of the weighted sequence $x_\alpha[n]$

* Relation Between DFT and DCT-2

$$\tilde{x}_2[n] = x[(n)_{2N}] + x[(-n-1)_{2N}]$$

$$X_2[k] = \underbrace{X[k]} + X^*[k]\, e^{j2\pi k/2N}$$

2N point DFT of N point signal $x[n]$

$$= e^{j\pi k/2N}\left[X[k].e^{-j\pi k/2N} + X^*[k]\, e^{j\pi k/2N}\right]$$

$$= e^{j\pi k/2N}.\underbrace{2.\mathrm{Re}\{X[k].e^{-j\pi k/2N}\}}$$

$$\Rightarrow \mathrm{Re}\left\{2.\sum_{n=0}^{2N-1} x[n].e^{-j2\pi kn/2N} . e^{-j\pi k/2N}\right\}$$

$$= \mathrm{Re}\left\{2.\sum_{n=0}^{2N-1} x[n].e^{-j\frac{\pi k(2n+1)}{2N}}\right\}$$

$$= 2.\sum_{n=0}^{N-1} x[n].\cos\left(\frac{\pi k(2n+1)}{2N}\right) = X^{c_2}[k]$$

$$\Rightarrow X_2[k] = e^{j\pi k/2N}.\, X^{c_2}[k] \quad , k = 0, 1 \dots N-1$$