Equiripple FIR Filter Design: (Eq-1)

* Purely Real desired freq. resp.
* Symmetric wrt $\omega=0$ Fourier Domain Specs $\Longrightarrow$

$$h_{id}[n] = h_{id}[-n]$$

$$H_{id}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_{id}[n] e^{-j\omega n} = h_{id}[0] + \sum_{n=1}^{\infty} 2h_{id}[n]\cos\omega n$$

$$H_d(e^{j\omega}) = \sum_{n=-L}^{L} h_L(n) e^{j\omega n} = h_L[0] + \sum_{n=1}^{L} 2h_L[n]\cos\omega n$$

design $\rightarrow d$ or L

$L+1$ filter order

$$H_d(e^{j\omega}) = \quad + \sum_{k=0}^{L} a_k (\cos\omega)^k = A(\omega)$$

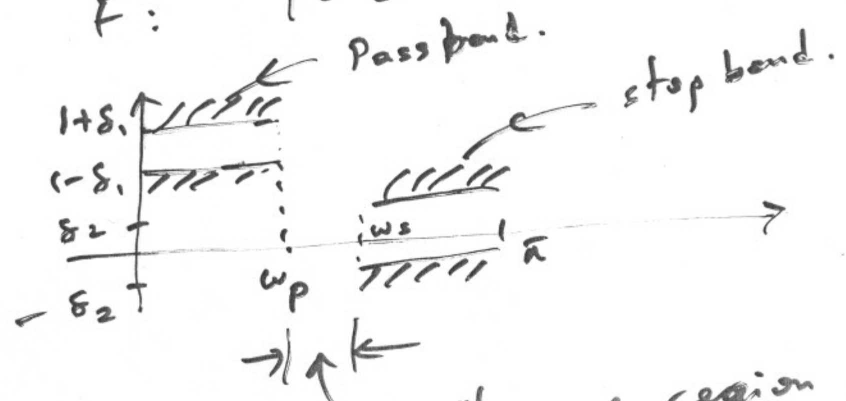
Using Chebyshev's relation $\quad T_n(\alpha) = \cos(n\cos^{-1}\alpha)$

$\Longrightarrow$ Use the techniques in polynomial approximation theory

Parks & McClellan solved the following problem: (minimize the maximum error):

$$\min_{h_L[n]} \left(\max_{\omega \in F} |E(\omega)| \right)$$

where F: $[0 \le \omega \le \omega_p] \cup [\omega_s \le \omega \le \pi]$



(remez)

$$E(\omega) = [H_{id}(e^{j\omega}) - A(\omega)]\, W(\omega)$$

$$W(\omega) = \begin{cases} \frac{\delta_2}{\delta_1} = \frac{1}{k}, & 0 \le \omega \le \omega_p \\ & \omega_s \le \omega \le \pi. \end{cases}$$

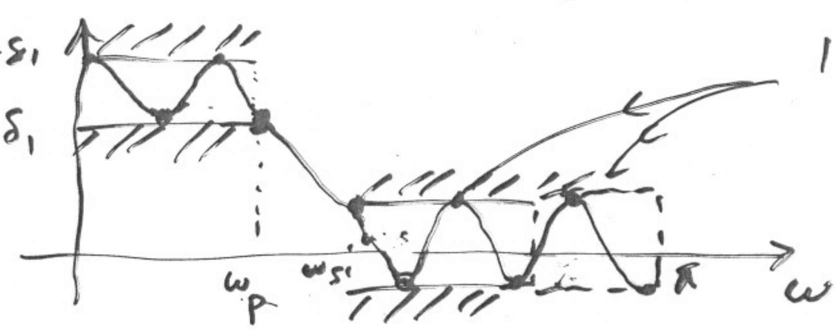
You have a complete control over the Freq. specs. (FIR - Filter Design Tool in MATLAB)

Remez : Specify (ω_p, ω_s, (filter order) or (δ_1 and δ_2)) $\rightarrow$ Solution.

ypical solutions: (Filter order ~~max 2L+1 = 7~~) ~~alternations~~:

max 2L+1+3 alternations

se 1)

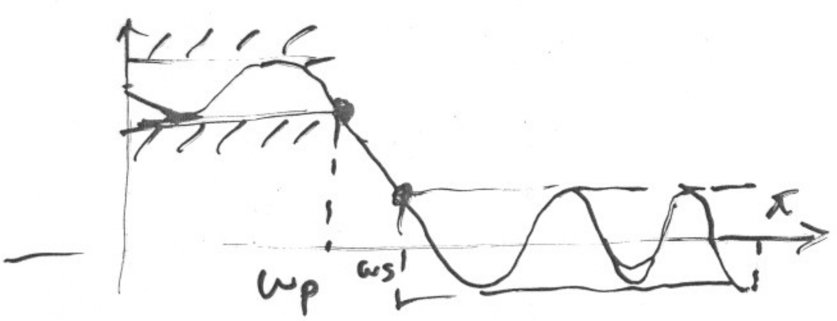


10 = L+3 alternations in the freq. resp.

extremum at π and 0. also ω_p and ω_s

se 2)

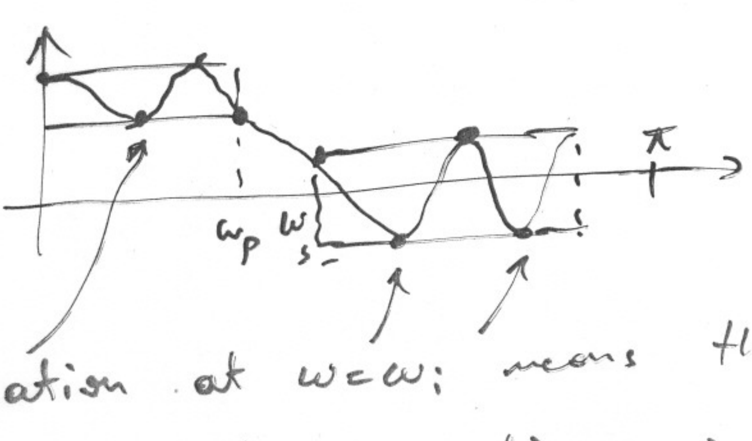
L+2 alterations:



extemum only at π ω_p, & ω_s

ase 3: Same as Case (2) but extrema only at $\omega = 0$ ω_p, ω_s

ase 4:



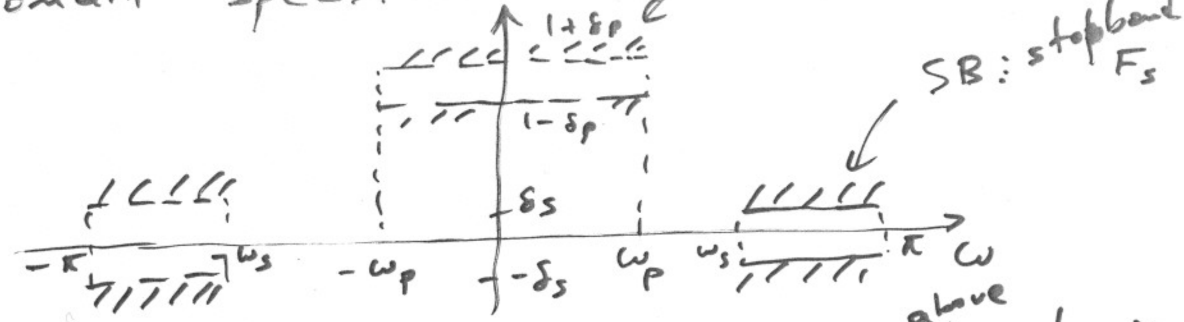
L+2 alteration with extrema at $\omega = 0$, ω_p, ω_s and $\omega = \pi$.

Alternation at $\omega = \omega_i$ means that $E_{\omega_i} = \max_{\omega \in F} E(\omega)$

The max # of alternations in a l.p. or highpass filter design is 2L+1+3. $\Rightarrow$ Equiripple FIR.

Equiripple FIR Filter Design by using the FFT based method.

Freq. Domain specs: PB : passband : F_p



* We want $H(e^{j\omega})$ to satisfy the above bounds in PB and SB.

  ↑ real

* Assume that $H(e^{j\omega}) = H(e^{-j\omega})$ (low-pass, high-pass, band-pass, band-stop)

$$\Rightarrow \quad h[n] = h[-n]$$

Time-domain constraint. (1.)

~~Freq. Domain Specs:~~

$$H_{id}(e^{j\omega}) - E_d(\omega) \leq H(e^{j\omega}) \leq H_{id}(e^{j\omega}) + E_d(\omega) \quad \text{for all } \omega$$

where

$$H_{id}(e^{j\omega}) = \begin{cases} 1 & \omega \in F_p \\ 0 & \omega \in F_s. \end{cases}$$

$$E_d(\omega) = \begin{cases} \delta_p & \omega \in F_p \\ \delta_s & \omega \in F_s \end{cases}$$

Time Domain Constraint (2)

$$h[n] = 0 \quad \text{for } n > L, \; n < -L \quad (\text{or } |n| > L)$$

Filter order is $2L+1$

(In the handout filter order is $2N+1$). (We use N for the FFT size so I'll use L for the filter size).

Iterative procedure: (Eq-2)

Initial step: $h_0[n]$ (Arbitrary but $h_0[n] = h_0[-n]$)

k-th iteration : $h_k[n]$

- Compute the DTFT of $h_k[n]$: $H_k(e^{j\omega})$ (Use FFT)
- Impose the freq. domain constraints on $H_k(e^{j\omega})$: (Freq. domain projection)

$$G_k(e^{j\omega}) = \begin{cases} H_{id}(e^{j\omega}) + E_d(\omega), & \text{if } H_k(e^{j\omega}) > H_{id}(e^{j\omega}) + E_d(\omega) \\ H_{id}(e^{j\omega}) - E_d(\omega), & \text{if } H_k(e^{j\omega}) < H_{id}(e^{j\omega}) - E_d(\omega) \\ H_k(e^{j\omega}), & \text{otherwise.} \end{cases}$$

- Compute the inverse F.T. of $G_k(e^{j\omega})$ and obtain $g_k[n]$. ($g_k[n] \equiv g_k[-n]$ real)

  (Use FFT. size $N \geq 10L$)

- Impose the time-domain constraint.
  " " projection:
  (FIR constraint)

$$h_{k+1}[n] = \begin{cases} g_k[n], & \text{for } n = 0, 1, \ldots, L \\ & \quad -1, -2, \ldots, -L \\ 0, & \text{otherwise.} \end{cases}$$

next iterate.

Repeat the above procedure with $h_{k+1}[n]$.
until $\|h_k[n] - h_{k+1}[n]\| < \varepsilon \leftarrow$ small number.

$\lim_{k \to \infty} h_k[n] =$ equiripple solution, if it exists.

* It gives the same solution as Parks-McClellan
... the lowest order FIR filter satisfying

Implementation Issues: (Eq-3)

1* Since $g_k[n]$ is an infinite extent signal in general FFT size N should be large. Higher N the better it is. e.g. $N > 10L$.

2* Make sure that ω_p and ω_s are in the FFT grid. $\left(\omega_k = \frac{2\pi k}{N},\ k = 0, 1, \dots N-1 \right)$.

3* Filter order parameter estimate: (lowpass & highpass)

$$L \cong \frac{-20 \log_{10} \sqrt{\delta_p \delta_s} - 13}{14.6\,(\omega_s - \omega_p)/2\pi} \qquad \text{(Kaiser)}$$

$\delta_p, \delta_s \downarrow$ filter order $L \uparrow$

$(\omega_s - \omega_p) \downarrow$ " " $L \uparrow$

Start the iterations with Kaiser's estimate.

4* How the algorithm works:

Set of freq. domain constraint.

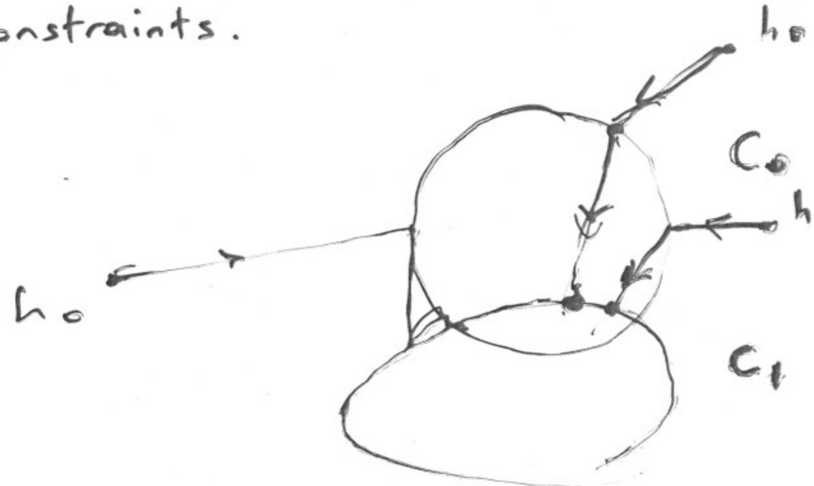
$C_0 \cap C_1 = \{ h^*$: equiripple solution

Time domain constraint set: Any member of this set is an FIR filter of order $2L+1$.

$\lim_{k \to \infty} h_k = h^*$

5* Too loose constraints (δ_p, δ_p may be too large or L may be too large). $\omega_s - \omega_p$ too large

We have many solutions satisfying the constraints.

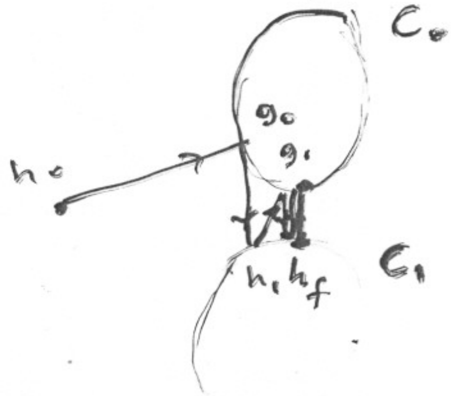


$C_0 \cap C_1$ contains many solutions

Result of the iterative algorithm will not be equiripple! $\Rightarrow$ either δ_p or $\delta_s \downarrow$ or $L \downarrow$ or $(\omega_s - \omega_p) \downarrow$

Start the iterations once again.

6* Too tight constraints: Too small L or to small δ_p, δ_s



No solution!

$\lim_{k \to \infty} h_k = h_f$ but $H_f(e^{j\omega})$ does not satisfy the freq. domain specs. $\Rightarrow$ $L \nearrow$ or δ_p, $\delta_s \nearrow$ ($|\omega_p - \omega_s| \nearrow$.)

7* Ex:11 $h[n] = \{\frac{1}{3}, \frac{1}{8}, \underset{\underset{n=0}{\uparrow}}{\frac{1}{2}}, \frac{1}{4}\} \xrightarrow{DFT} H[k] = ?$

Everything is periodic in DFT implementation.

Take the DFT of $\tilde{h}[n] = \{\underset{\underset{n=0}{\uparrow}}{\frac{1}{2}}, \frac{1}{4}, 0, \dots, \frac{1}{3}, \underset{\underset{n=N-1}{\uparrow}}{\frac{1}{8}}\}$

$$\tilde{H}[k] = H[k]$$

7*b shift the anti causal $h[n]$ so that it is causal then compute the DFT.

because $\tilde{h}[k] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{H}[k] e^{j\frac{2\pi kn}{N}}$ is periodic with period N.

$\Rightarrow \quad \tilde{h}[n] = h[n] \quad$ for $n = -\frac{N}{2}, \dots, 0, \dots, \frac{N}{2} - 1$

Example
Time domain projection of $g_k = [a, b, c, 0, 0 \dots, 0, c, b]$ FIR filter] is for $L = 1$ (meaning a 3rd order

$$h_{k+1}[n] = [a, b, 0, 0, \dots, 0, b].$$

8* Translate the Freq. Domain specs from $[-\pi, \pi]$ to $w \in [0, 2\pi]$ or $[0, 1, \dots, N-1]$ in the DFT domain!