IIR Filter Design (Cont'd)

11 - March 26

Butterworth Filters:

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(j\Omega/j\Omega_c\right)^{2N}}, \quad H_a(s)H_a(-s) = \frac{1}{1+\left(s/j\Omega_c\right)^{2N}}$$

lowpass analog filter with cut off at Ω_c.

select the poles in the l.h.p.

Discrete-time filter

$$H_a(s) \xrightarrow[\text{bilinear transf.}]{} H(z)$$

$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

poles in the unit circle.

Ex 1)

p.b. $\quad 1 \geq |H(e^{j\omega})| \geq 0.89125 \; (-1\,dB) \qquad 0 \leq \omega \leq 0.2\pi.$

s.b. $\quad |H(e^{j\omega})| \leq 0.17783 \; (-15\,dB) \qquad 0.3\pi \leq |\omega| \leq \pi$

Use bilinear transformation to design the filter and Butterworth filter

actual freq. $\rightarrow \Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$ ← normalized freq. used in D.T. domain.

$$1 \geq |H_a(j\Omega)| \geq 0.89125 \qquad 0 \leq \Omega \leq \frac{2}{T}\tan\left(\frac{0.2\pi}{2}\right)$$

$$|H_a(j\Omega)| \leq 0.17783, \quad \frac{2}{T}\tan\left(\frac{0.3\pi}{2}\right) \leq \Omega \leq \infty$$

Let $T = 1$. $\quad |H(j\Omega)| = \sqrt{\dfrac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}}$

$$1+\left(\frac{2\tan 0.1\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89}\right)^2$$

$$(**) \quad 1+\left(\frac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.178}\right)^2$$

$N = 5.3$ or 5.6

$\Rightarrow$ N must be an integer

$N = 6$

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$$N = \log\left[\left(\left(\frac{1}{0.17}\right)^2 - 1\right) \Big/ \left(\left(\frac{1}{0.89}\right)^2 - 1\right)\right] \Big/ 2\log\left[\tan 0.15\pi / \tan 0.1\pi\right]$$

$N = 5.3046\ldots$

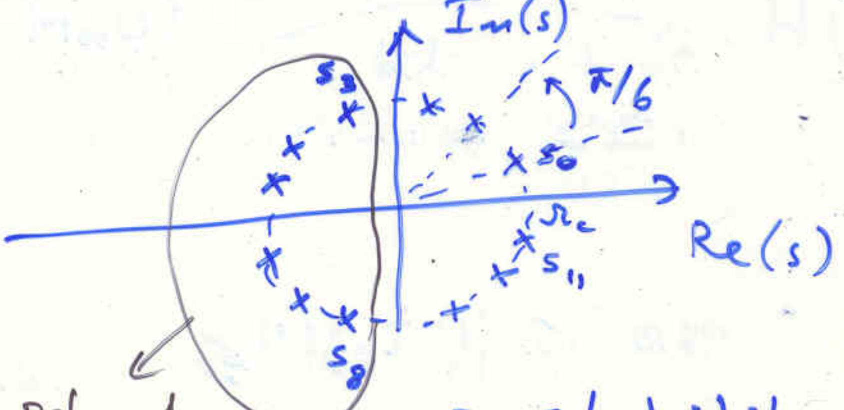
Substitute $N=6$ to (**) to find

$\Omega_c = 0.7662$.

Analog prototype:

$$H_a(s)\, H_a(-s) = \frac{1}{1 + \left(\frac{s}{j0.7662}\right)^{12}}$$

12 poles.



$$s_k = \Omega_c\, e^{j\frac{\pi}{12}(2k+N-1)}$$

$k = 0, 1, 2, \ldots, 11$

poles of H(s) $\Rightarrow$ stability.

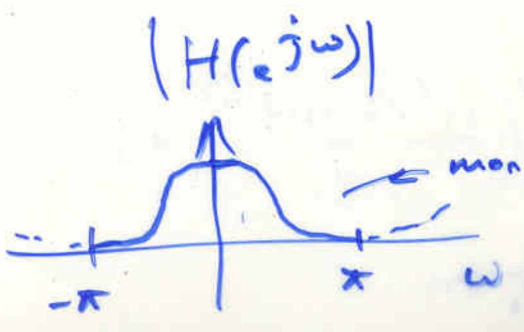
$$H(s) = \frac{C}{(s-s_3)(s-s_4)\cdots(s-s_8)}$$

$H(j0) = 1 \Rightarrow$ gives you C.

Discrete-time IIR filter:

$$H(z) = H(s)\Big|_{s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$|H(e^{j\omega})|$



monotonically decreasing in $[0, \pi]$.

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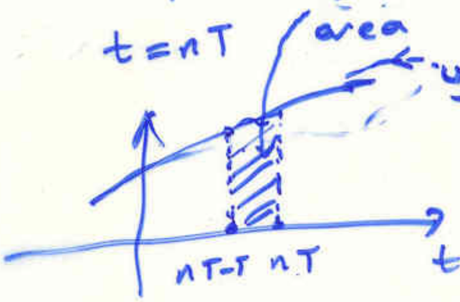
IIR Filte Design by Bilinea Transform.

* Use the trapezoidal rule. for integration.

$$\frac{dy(t)}{dt} + \alpha\, y(t) = x(t) \qquad (1)$$

$$y(t) = \int_{nT-T}^{nT} y'(\tau)\, d\tau + y(nT-T) \qquad (*)$$

where $y'(t)$ is the derivative.



$$y(t) = \int_{-\infty}^{t} y'(\tau)\, d\tau$$

$$y(t) = \int_{t_0}^{t} y'(\tau)\, d\tau + \int_{-\infty}^{t_0} y'(\tau)\, d\tau$$

$$y(t) = \int_{t_0}^{t} y'(\tau)\, d\tau + y(t_0)$$

$$t_0 = nT - T$$

To approximate the area use trapezoidal approx.

$$\text{Area} \cong \frac{T}{2}\left(y'(nT) + y'(nT-T)\right)$$

Equation (*) can be approximated as

$$y(nT) = y[n] \cong \frac{T}{2}\left(y'[n] + y'[n-1]\right) + y[n-1]$$

From (1)

$$y'[n] = y'(nT) = -\alpha\, y(nT) + x(nT)$$

$$y'[n-1] = y'(nT-T) = -\alpha\, y(nT-T) + x(nT-T)$$

Corresponding difference equation. March 26-5

$$y[n] = \frac{T}{2}\left(\underbrace{-\alpha y[n] + x[n]}_{y'[n]} \underbrace{-\alpha y[n-1] + x[n-1]}_{y'[n-1]}\right) + y[n-1]$$

In z-domain

$$\left(1+\frac{\alpha T}{2}\right)Y(z) - \left(1-\frac{\alpha T}{2}\right)Y(z)z^{-1} = \frac{T}{2}\left(1+z^{-1}\right)X(z)$$

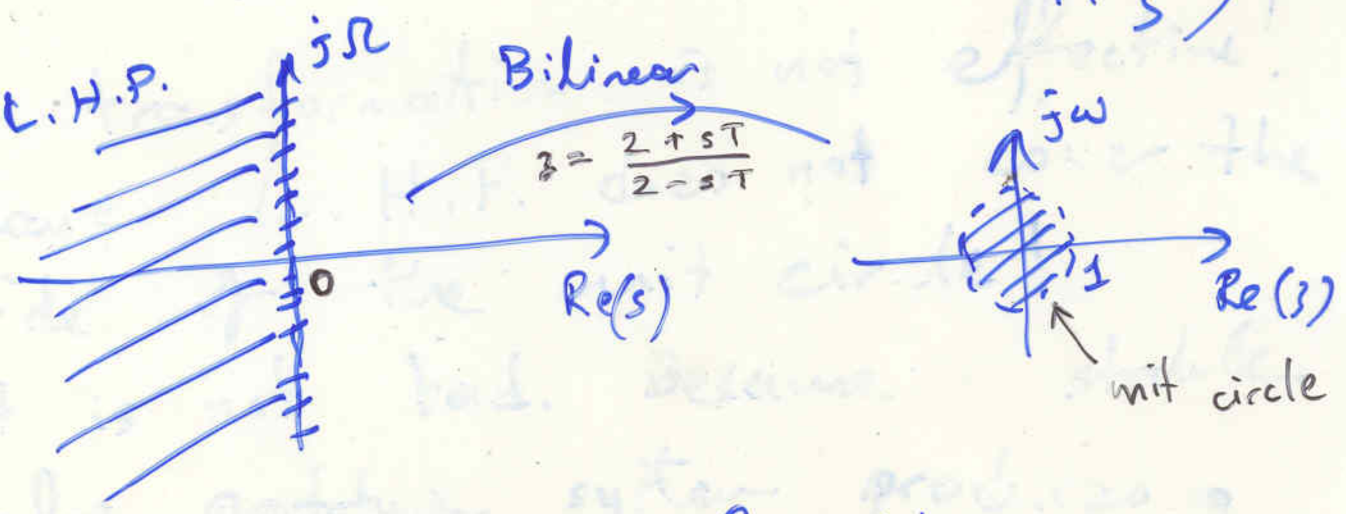
or $$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\alpha}$$

(Analog system $H_a(s) = \frac{1}{s+\alpha}$)

We replace s by the so-called Bilinear transformation: $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ to obtain $H(z)$ from $H_a(s)$.

* Given an analog prototype $H_a(s)$
Obtain the discrete-time filter:

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$



* Efficient transformation.

* stable analog filter $\Longrightarrow$ digital stable filter.