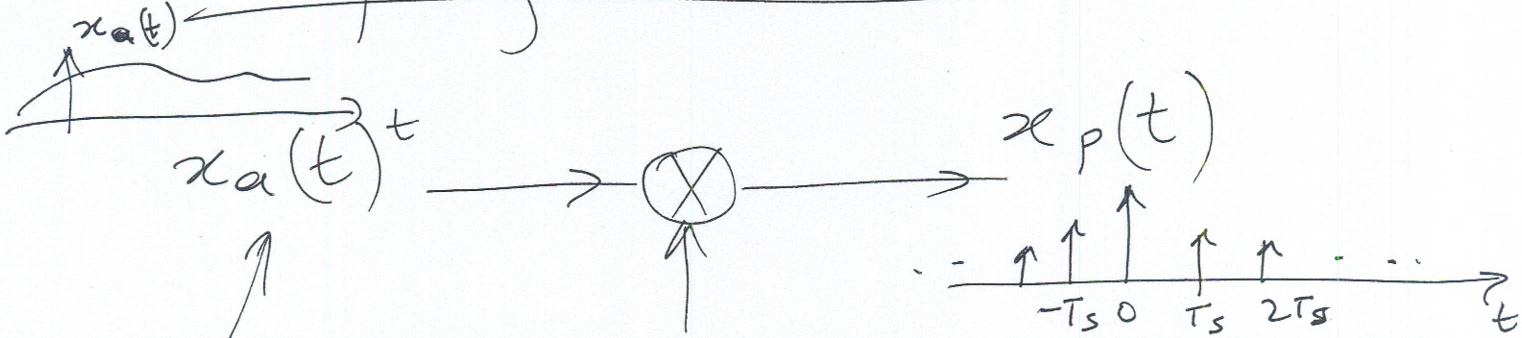
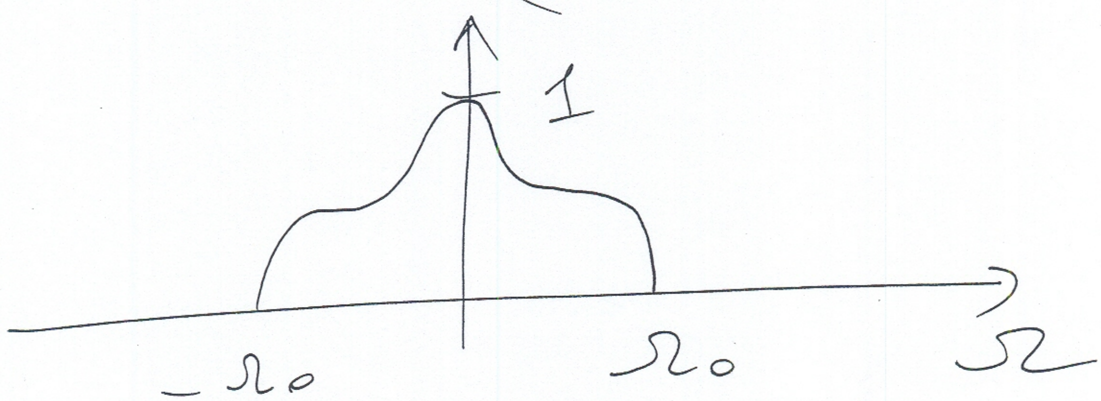
Sampling Theorem:



band limited

$$p(t) = \sum_n \delta(t - nT_s)$$

$X_a(j\Omega)$



$$p(t) \xleftrightarrow{\mathcal{F}_{CT}} P(j\Omega)$$

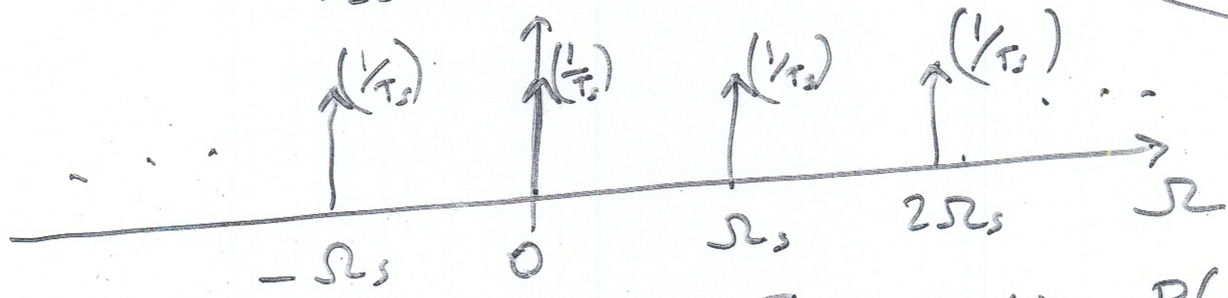
1. Sampling Theorem

$x_a(t) \longrightarrow \otimes \longrightarrow p(t)$

$\uparrow p(t) = \sum_n \delta(t - nT_s)$

$\Omega_s = \frac{2\pi}{T_s}$

$$P(j\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$



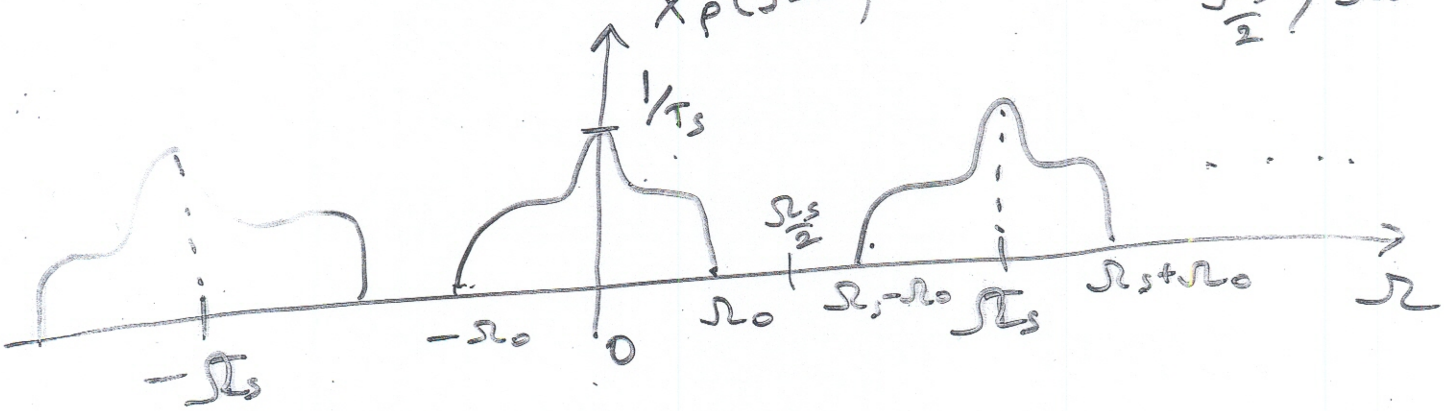
$$x_p(t) = p(t)\cdot x_a(t) \xleftrightarrow{F_{CT}} X_a(j\Omega) * P(j\Omega)$$

$$X_p(j\Omega) = X_a(j\Omega) * P(j\Omega)$$
$$= X_a(j\Omega) * \frac{1}{T_s}\sum_k \delta(\Omega - k\Omega_s)$$

$$\left(\delta(\Omega - k\Omega_s) * X_a(j\Omega) = X_a(j(\Omega - k\Omega_s))\right)$$

$$X_p(j\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_s))$$

$\frac{\Omega_s}{2} > \Omega_0$



No aliasing: $\Omega_s - \Omega_0 > \Omega_0$

Nyquist rate $\Omega_s > 2\Omega_0$ $\quad (f_s > 2f_0)$

Shannon: You can reconstruct the original signal (band-limited) from its samples using a low-pass filter, iff $\Omega_s > 2\Omega_0$.

② Cont-time F.T. of $x_p(t)$:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\, \delta(t - nT_s)$$

$$X_p(j\Omega) = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_a(nT_s)\, \delta(t-nT_s) \right) e^{-j\Omega t}\, dt$$

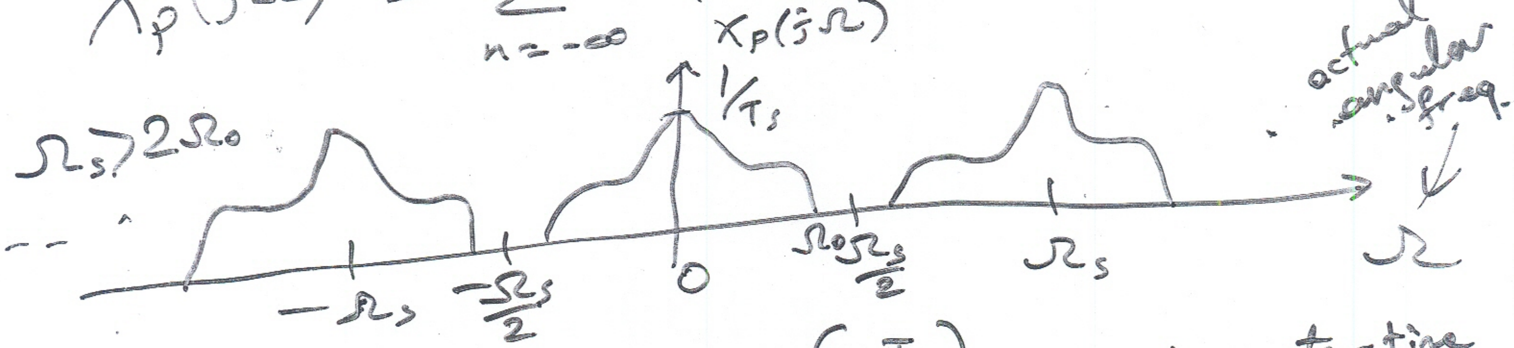
$$= \sum_{n=-\infty}^{\infty} x_a(nT_s) \int_{-\infty}^{\infty} \delta(t-nT_s)\, e^{-j\Omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT_s) \int_{-\infty}^{\infty} \delta(t-nT_s)\, e^{-j\Omega n T_s}\, dt$$

$$X_p(j\Omega) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j\Omega T_s n} \underbrace{\int_{-\infty}^{\infty} \delta(t-nT_s)\, dt}_{1}$$

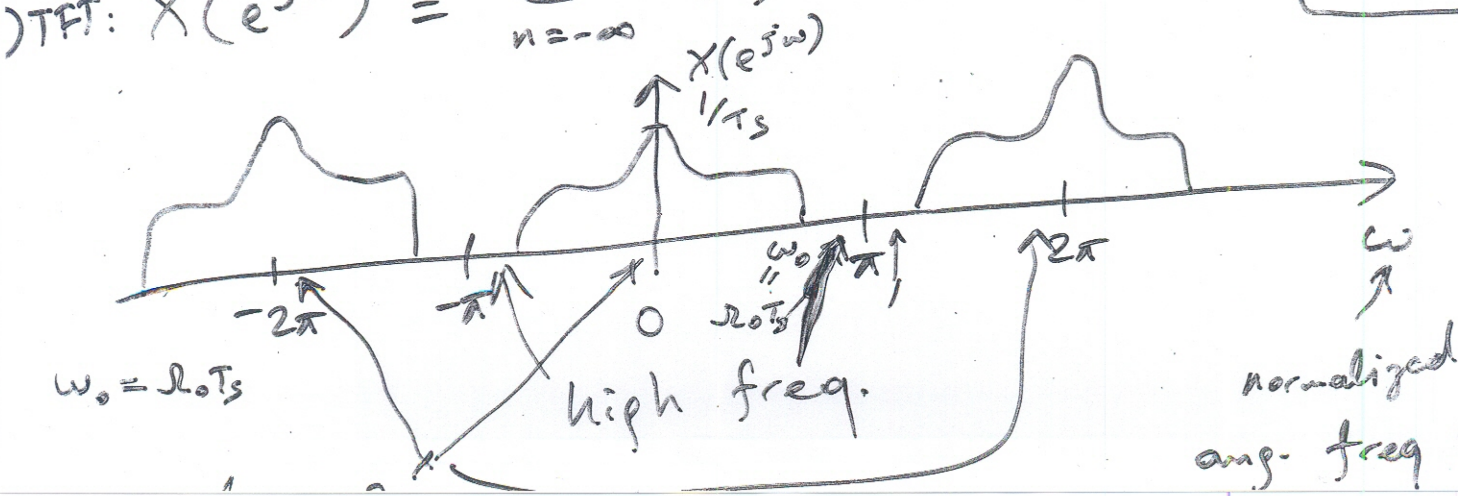
$$X_p(j\Omega) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j\Omega T_s n} \qquad (1)$$

$\Omega_s > 2\Omega_0$



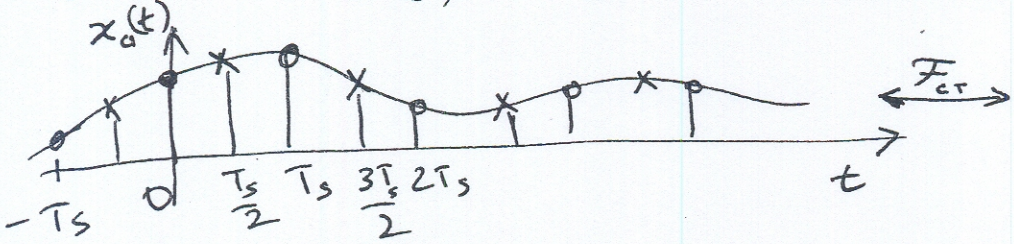
$x_p(t) \equiv \underbrace{x[n] = x(nT_s)}$ ← Discrete-time signal

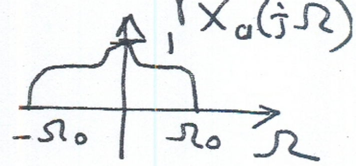
DTFT: $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad (2) \qquad \boxed{\omega = \Omega T_s}$$



Interpolation:

Let $x_a(t)$ be a bandlimited signal





$x[n] = x_c(nT_s)$: "•" : sampled with Ω

$x_i[n] = x_c\left(n\frac{T_s}{2}\right)$: "•" & "x" : sampled with $2\Omega_s$. $(\Omega_s > 2\Omega_0 \Rightarrow 2\Omega_s > 2\Omega$

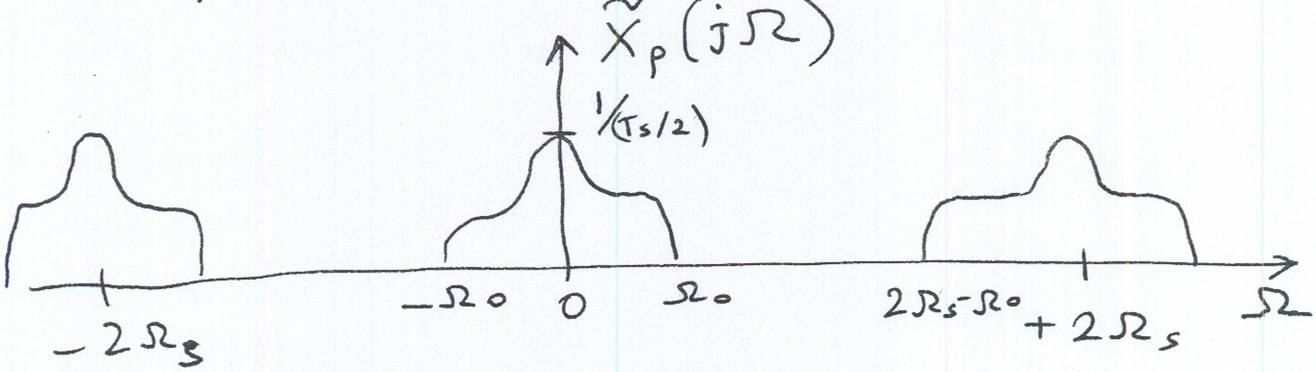
* Interpolation problem:
Obtain $x_i[n]$ from $x[n]$ in discrete time domain.

* Sampling the analog signal $x_c(t)$ with $2\Omega_s$:

$$x_a(t) \longrightarrow \otimes \longrightarrow \tilde{x}_p(t) \equiv x_i[n]$$

equivalent

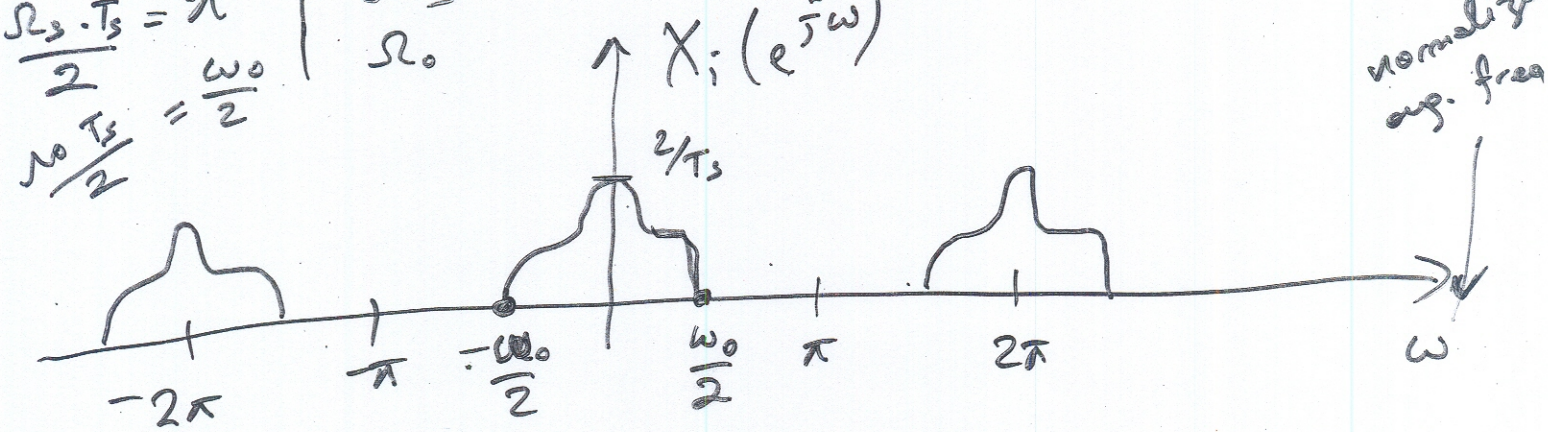
$$\uparrow \sum_{n=-\infty}^{\infty} \delta\left(t - n\frac{T_s}{2}\right)$$



The relation between $\tilde{X}_p(j\Omega)$ & $X_i(e^{j\omega})$ $\quad \omega = \Omega \frac{T_s}{2}$

where $X_i(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_i[n] e^{-j\omega n}$

ω	Ω
0	0
2π	$2\Omega_s$
$\frac{\Omega_s \cdot T_s}{2} = \pi$	Ω_s
$\frac{\Omega_0 T_s}{2} = \frac{\omega_0}{2}$	Ω_0



When the sampling freq. is $2\Omega_s$ the highest normalized freq π corresponds to Ω_s.

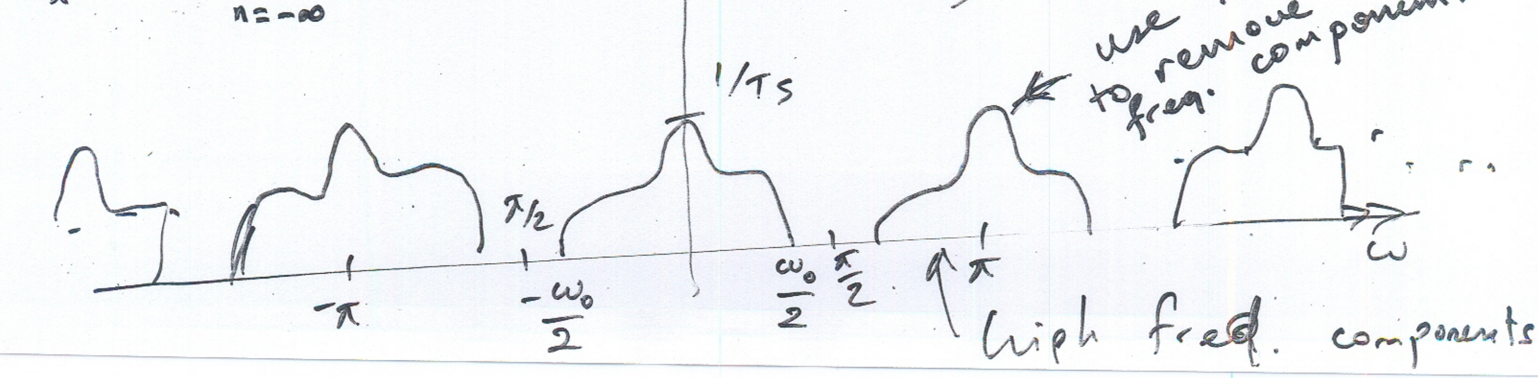
$X(e^{j\omega})$ $(x[n])$ contains all the necessary information to obtain $X_i(e^{j\omega})$ $(x_i[n])$

$X_u(e^{j\omega})$: period π

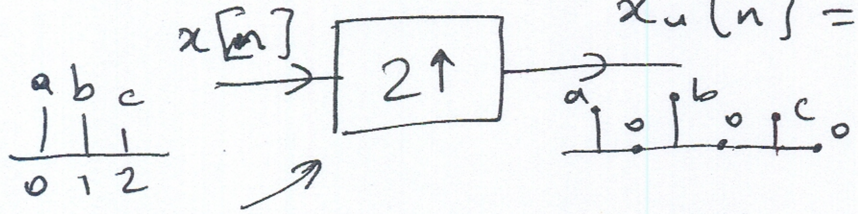
$= X(e^{j2\omega})$

$x[n] \rightarrow \boxed{2\uparrow} \rightarrow x_u[n]$

$X_u(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\omega n}$



Upsampler:



$$x_u[n] = \begin{cases} x[n/2] & , n \text{ even} \\ 0 & , n \text{ odd.} \end{cases}$$

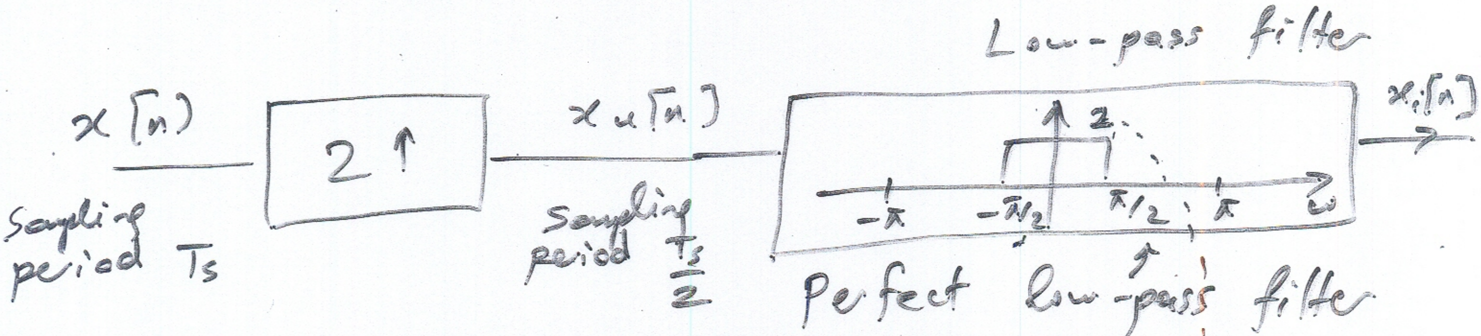
Insert a zero between every-other sample

$$X_u(e^{j\omega}) = \overset{x[0]}{\overset{\shortparallel}{x_u[0]}} + \overset{0}{\overset{\shortparallel}{x_u[1]}}\, e^{-j\omega} + \overset{x[1]}{\overset{\shortparallel}{x_u[2]}}\, e^{-j2\omega} + \cdots$$
$$+ \underset{0}{\underset{\shortparallel}{x_u[-1]}}\, e^{+j\omega} + \underset{x[-1]}{\underset{\shortparallel}{x_u[-2]}}\, e^{j2\omega} + \cdots$$

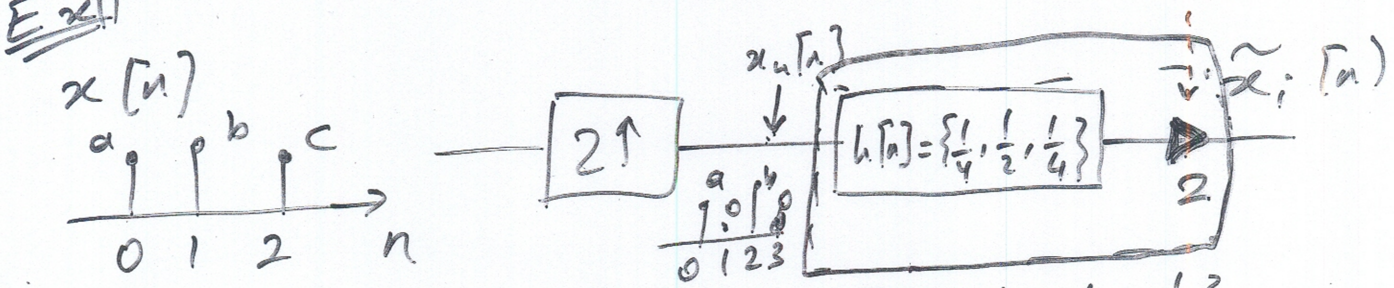
$$X_u(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\omega n}$$

$$X_u(e^{j\omega}) = X(e^{j2\omega})$$

Interpolation (by a factor of 2)

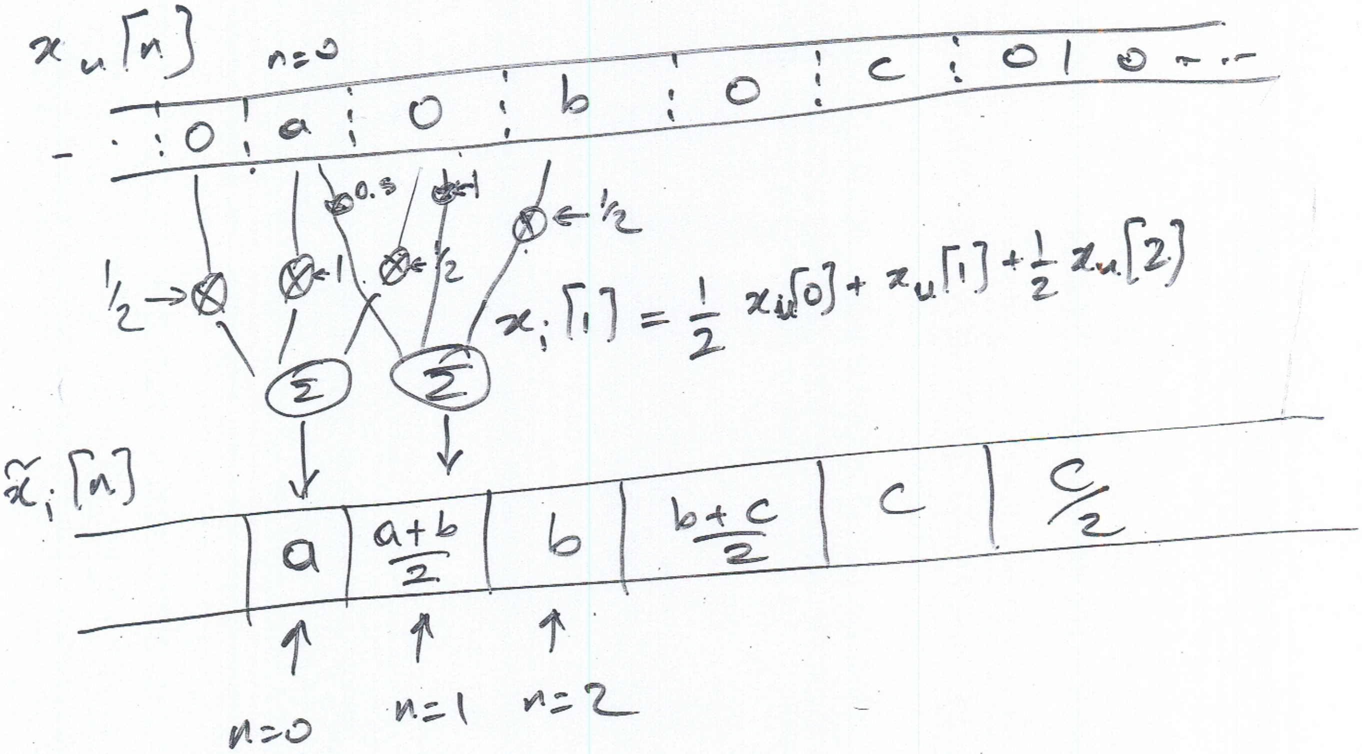


Ex 11



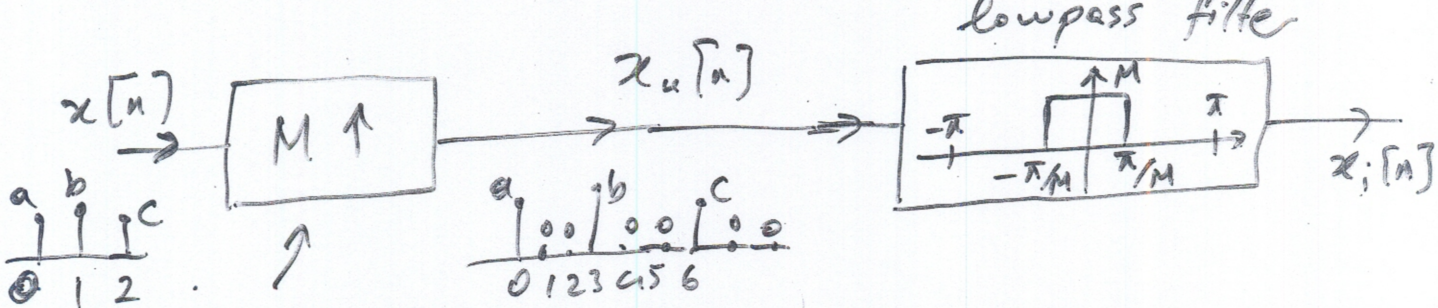
$g[n]=\{\frac{1}{2}, 1, \frac{1}{2}\}$

$\left(G(e^{j\omega}) = 1 + 1\cos\omega\right)$



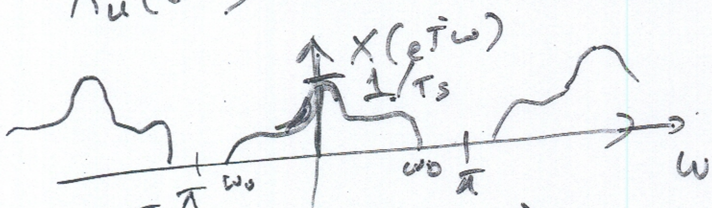
$x_i[1] = \frac{1}{2} x_u[0] + x_u[1] + \frac{1}{2} x_u[2]$

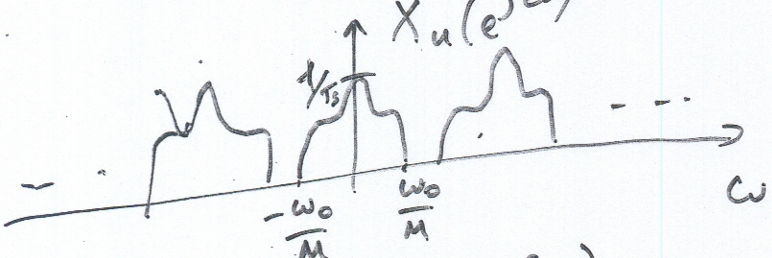
Interpolation by a factor of M!

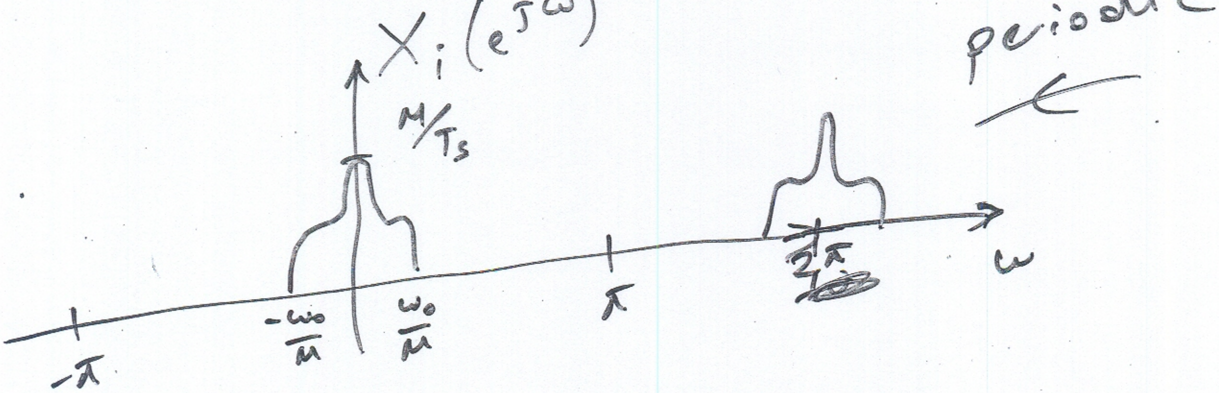


Insert M-1 zeros between every othe sample of x[n] (M=3)

$$X_u(e^{j\omega}) = X(e^{jM\omega})$$

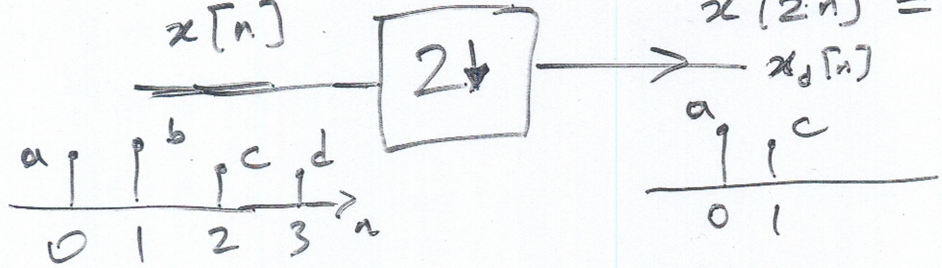






Downsampling:

$x[2n] = x_d[n]$



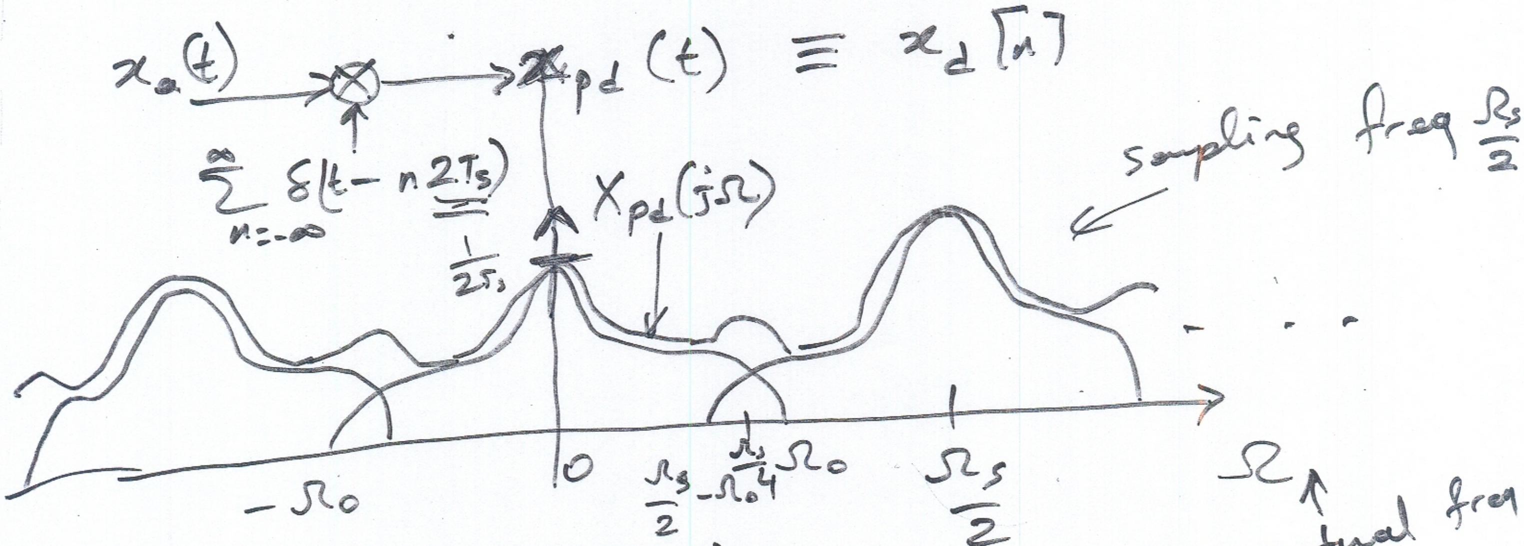
$x_d[0] = x[0]$
$x_d[1] = x[2]$
$x_d[2] = x[4]$.

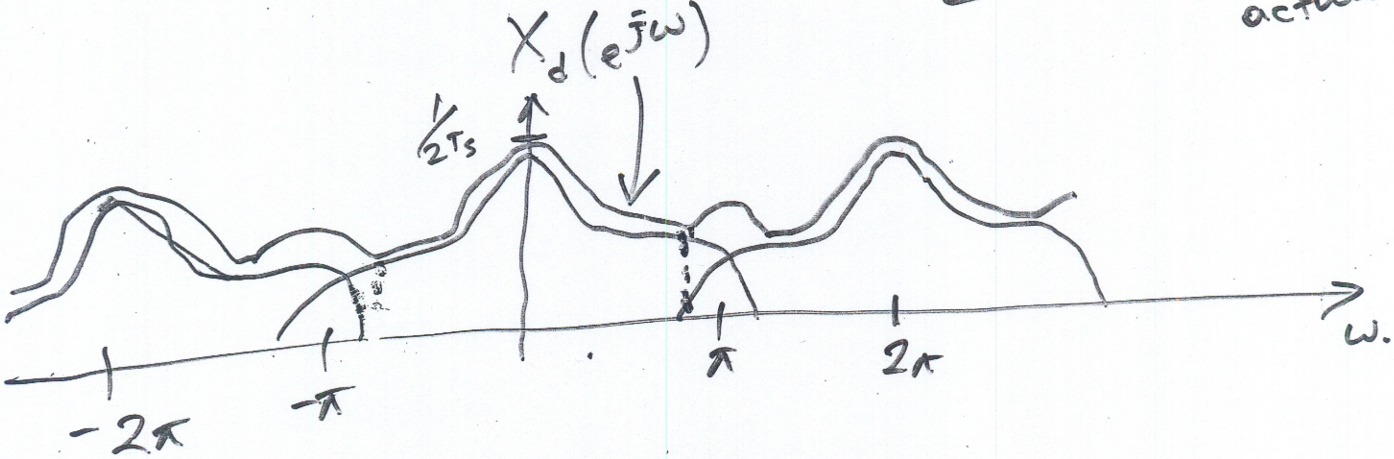
Actual sampling period: T_s
f_s, $\Omega_s = 2\pi f_s$

~~Actual~~ sampling period : $2T_s$, $\frac{f_s}{2}$, $\frac{\Omega_s}{2}$

(Sampling Theorem : $\Omega_s > 2\Omega_0$)

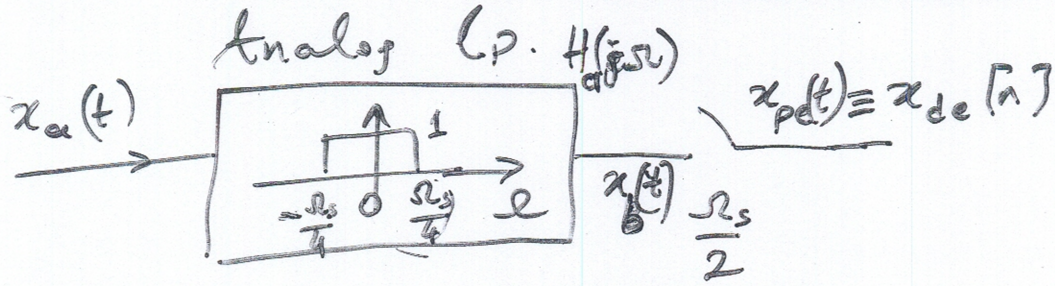
If $\Omega_0 > \frac{\Omega_s}{2}$ then we may have aliasing : $\Longrightarrow$

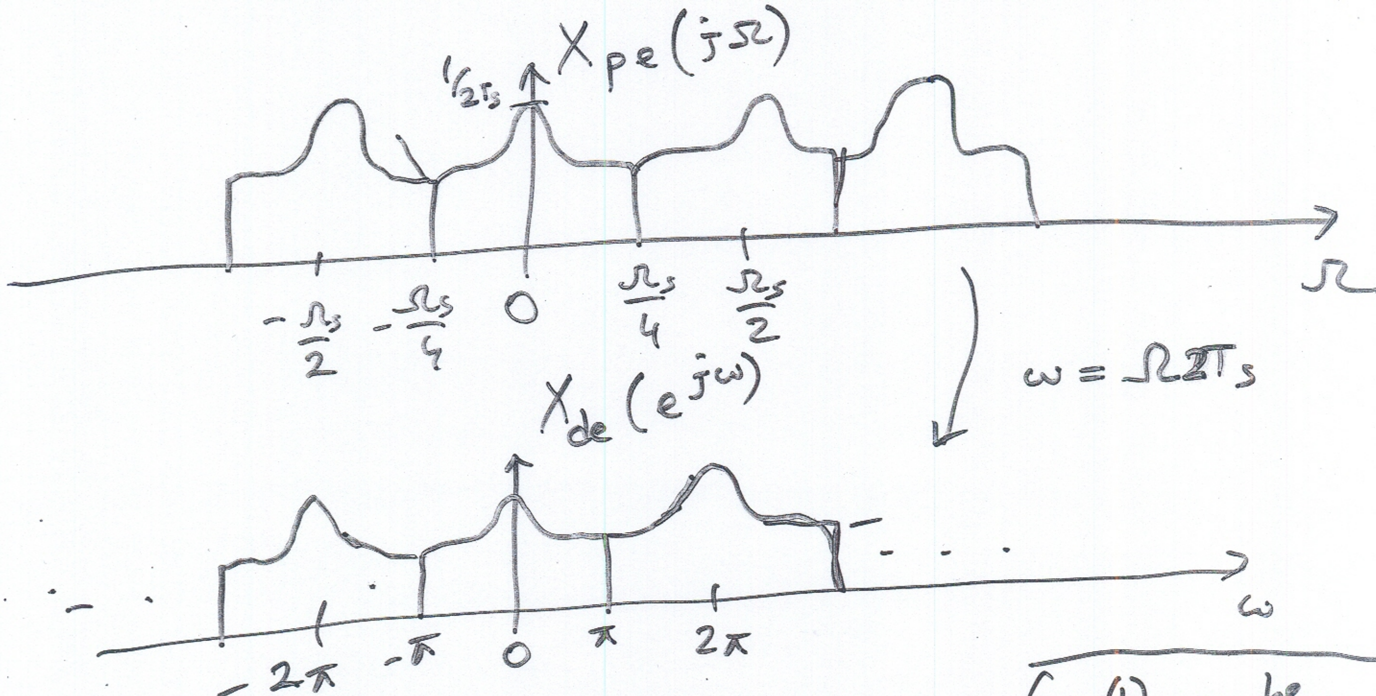


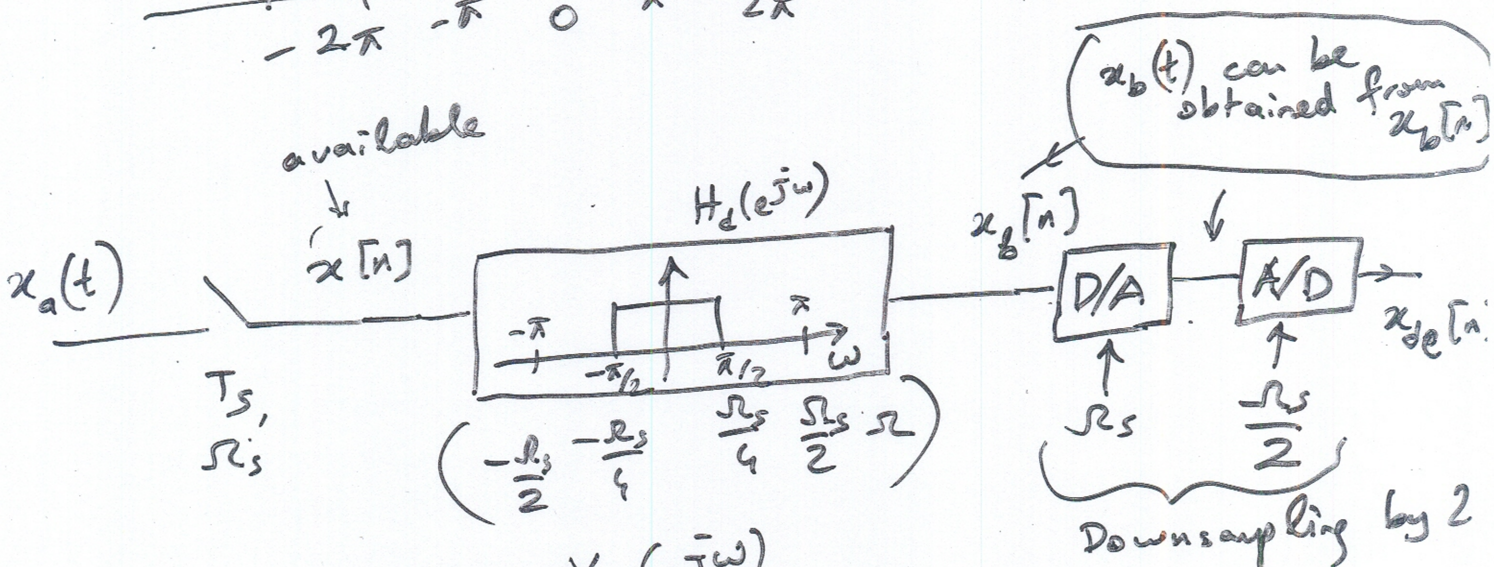


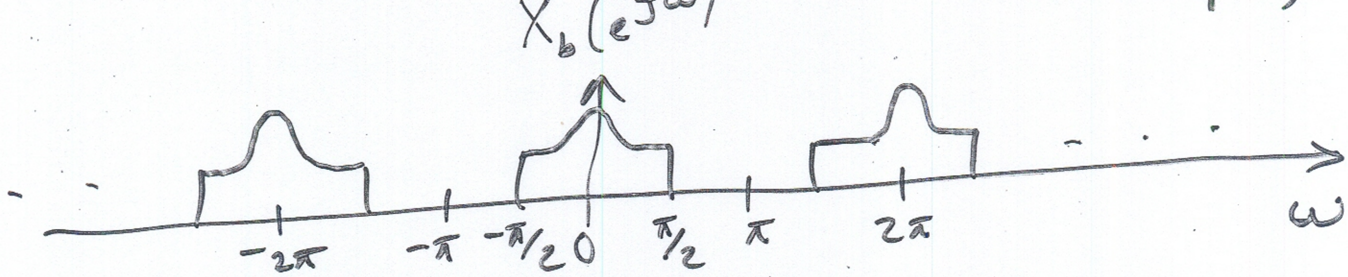
When $\Omega_0 > \frac{\Omega_s}{4}$ there must be aliasing!

You cannot simply throw away samples!

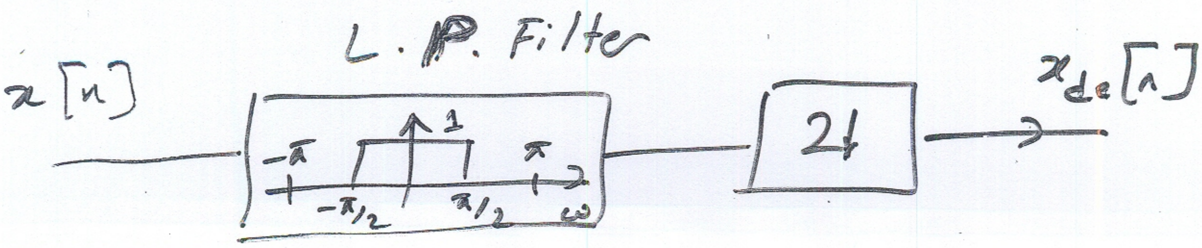


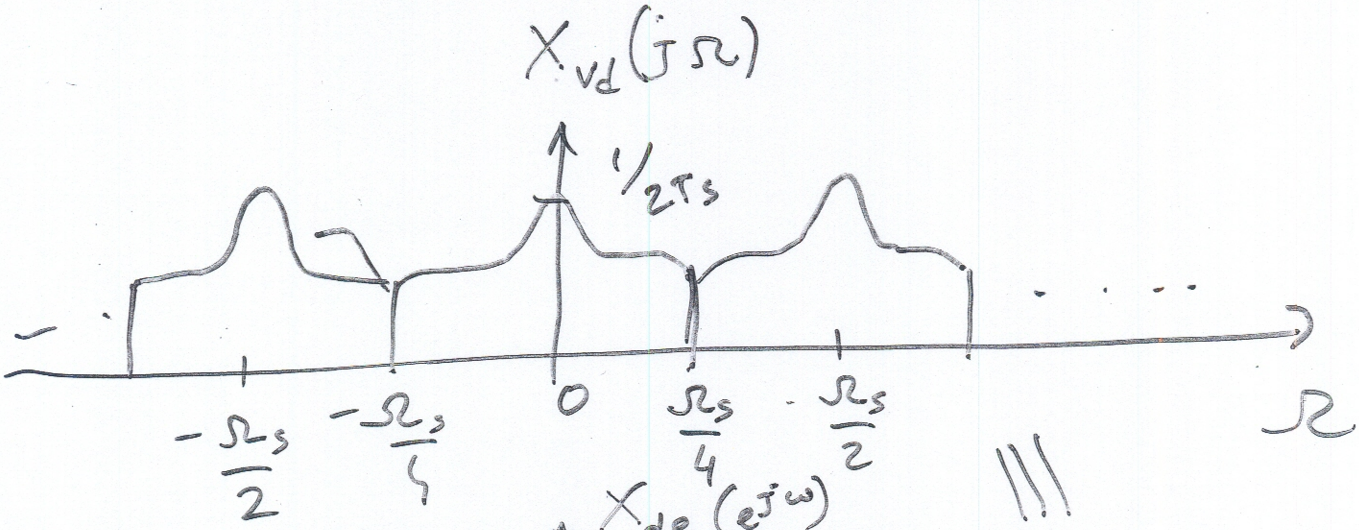


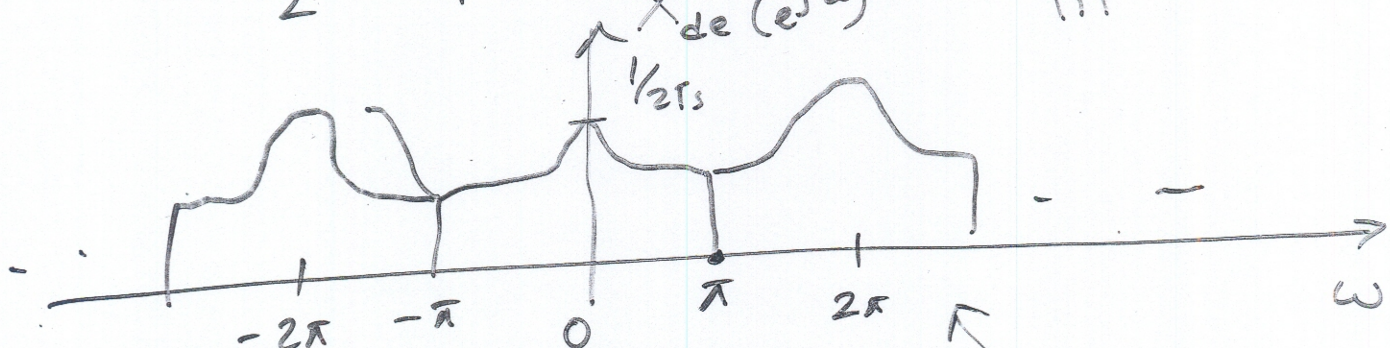




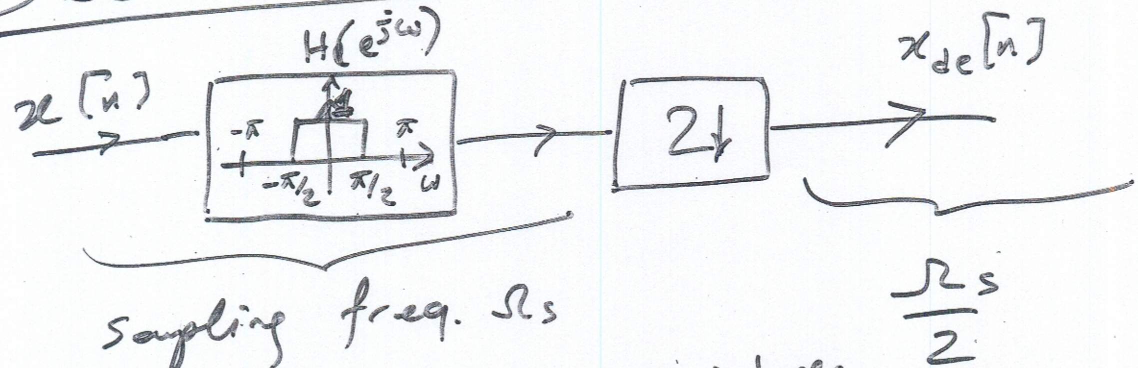
Decimation: by $M = 2$



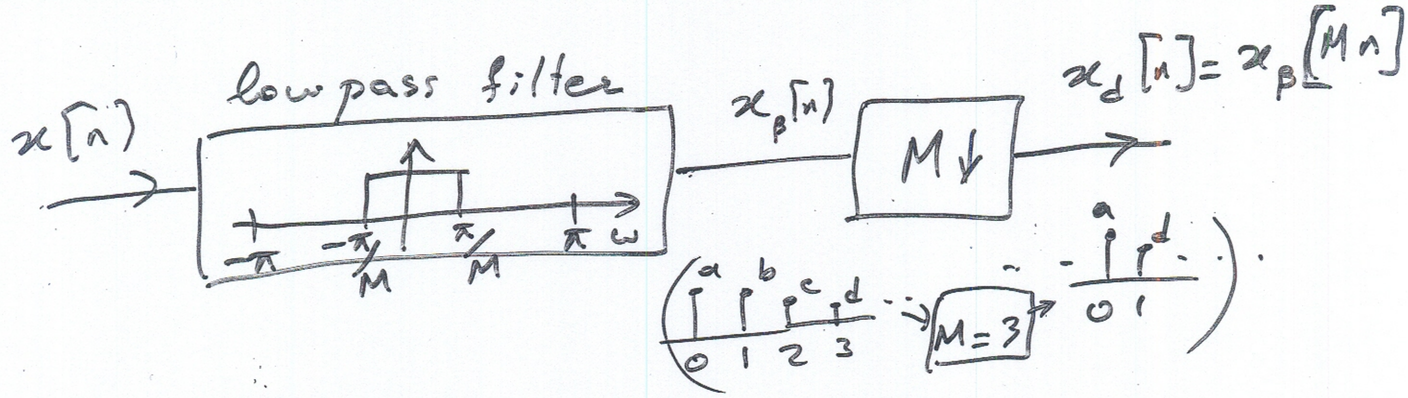




Decimation: low-pass + downsampling (by 2).



Decimation by M: ← integer



* Decimation, in general, is a lossy operation!

Ex $h[n] = \{\underset{n=0}{\frac{1}{4}}, \frac{1}{2}, \frac{1}{4}\}$ ← Causal

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{2} h[k] e^{-j\hat{\omega} k} = \frac{1}{4}\underbrace{}_{e^{-j\hat{\omega}0}} + \frac{1}{2} e^{-j\hat{\omega}\cdot 1} + \frac{1}{4} e^{-j\hat{\omega}\cdot 2}$$

$$= e^{-j\hat{\omega}}\left(\frac{1}{4} e^{j\hat{\omega}} + \frac{1}{2} + \frac{1}{4} e^{-j\hat{\omega}}\right)$$

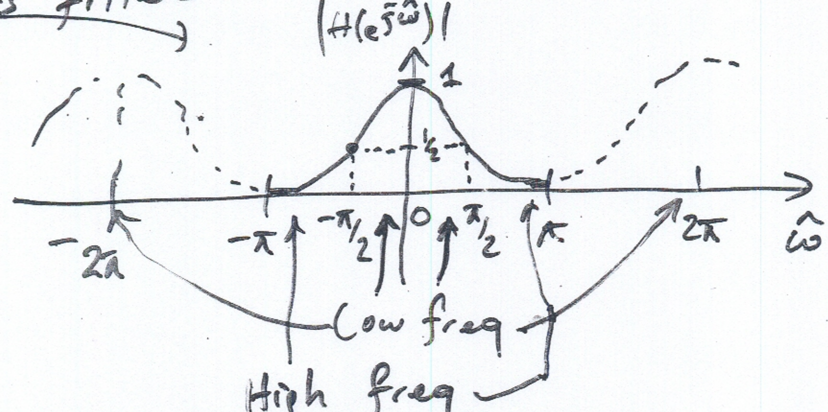
$$H(e^{j\hat{\omega}}) = \left(\frac{1}{2} + \frac{1}{2}\cos\hat{\omega}\right) e^{-j\hat{\omega}}$$

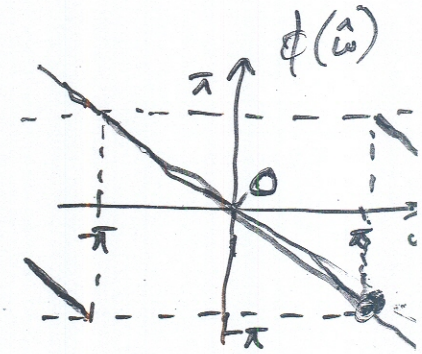
Magnitude response $\quad |H(e^{j\hat{\omega}})| = \left|\frac{1}{2} + \frac{1}{2}\cos\hat{\omega}\right| \underbrace{|e^{-j\hat{\omega}}|}_{=1}$

$$|H(e^{j\hat{\omega}})| = \frac{1}{2} + \frac{1}{2}\cos\hat{\omega}$$

Phase response : $\quad \phi(\hat{\omega}) = -\hat{\omega}$

Low-pass filter





$\phi(3\pi = \pi)_{\text{mod } 2\pi}$

$\hat{\phi}(3\pi) = -3\pi$

$\hat{\phi}(3\pi) = -\pi = +\pi$

This low-pass filter can be used in interpolation & decimation!

Theorem: $H(\hat{\omega})$ is 2π periodic.

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{N-1} h[k] e^{-j\hat{\omega} k} \overset{?}{=} H(e^{j(\hat{\omega} + 2\pi L)}) \quad (+2\pi L \leftarrow \text{integer})$$

$$\overset{?}{=} \sum_{k=0}^{N-1} h[k] e^{-j(\hat{\omega} + 2\pi L) k}$$

$$= \sum_{k=0}^{N-1} h[k] e^{-j\hat{\omega} k} \cdot \underbrace{e^{-j2\pi L}}_{=1}$$

$$\underline{H(e^{j\hat{\omega}}) = H(e^{j(\hat{\omega} + 2\pi L)})}$$

Ex $n = 0, \mp 1, \mp 2, \dots$

L.T.I
Freq. resp.

$\cos\hat{\omega}_0 n$ → $\boxed{H(e^{j\hat{\omega}})}$ → $y[n] = ?$

$\frac{1}{2} e^{j\hat{\omega}_0 n} + \frac{1}{2} e^{-j\hat{\omega}_0 n}$