# Circular Convolution ~~and~~ Convolution using DFT

$$x_1[n] \xleftrightarrow{DFT_N} X_1[k] \quad , k = 0, 1, \ldots, N-1$$
$$x_2[n] \xleftrightarrow{DFT_N} X_2[k] \quad , k = 0, 1, \ldots, N-1.$$

Define $X_3[k] = X_1[k] \cdot X_2[k]$

$$\Big\updownarrow IDFT_N$$

$$x_3[n] = \sum_{m=0}^{N-1} x_1[n]\, x_2[\langle n-m \rangle_N] \quad , n = 0, 1, \ldots, N-1$$

$\Rightarrow$ Not an ordinary convolution!

$$X_{1,2}[k] = X_{1,2}(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k} \quad , k = 0, 1, \ldots, N-1$$

because $x_1[n]$ is a finite-extent seq.

$$X_3[k] = \left(X_1(e^{j\omega})\, X_2(e^{j\omega})\right)\Big|_{\omega = \frac{2\pi}{N}k} \quad , k = 0, 1, \ldots, N-1$$

Length of $y[n] = (x_1 * x_2)[n]$ is $L_1 + L_2 - 1$

If $N \geq L_1 + L_2 - 1$. then

$$Y[k] = Y(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k} = \left(X_1(e^{j\omega})\, X_2(e^{j\omega})\right)\Big|_{\omega = \frac{2\pi}{N}k}$$

$$Y[k] = X_3[k] \quad , k = 0, 1, \ldots, N-1$$

and $y[n] = x_3[n]$.

* This property may be used to compute convolution!

Ex.|| $x_d[n] = \begin{cases} 1, & n=1 \\ 0, & n=0,2,3,4 \ldots N-1 \end{cases} = \delta_N[n-1]$

Calculate $X_d[k] = ?$

$X_d[0] = 1$.

$X_d[1] = \sum_{n=0}^{N-1} x_d[n] e^{-j\frac{2\pi}{N}kn} = x_d[1] e^{-j\frac{2\pi}{N}\cdot 1 \cdot 1} = e^{-j\frac{2\pi}{N}}$

$X_d[l] = \sum_{n=0}^{N-1} x_d[n] e^{-j\frac{2\pi}{N}ln} = x_d[1] e^{-j\frac{2\pi}{N}l\cdot 1} = e^{-j\frac{2\pi}{N}l}$

$l = 0, 1, \ldots, N-1$

If $x_d[n] = x[n-l]$

$0 < l < N-1$

some of the samples may leave the index range $n = 0, 1, \ldots, N-1$.

So conside a periodic shift:

$x_d[n] \triangleq x[\langle n - n_0 \rangle_M]$ $\quad (\langle \cdot \rangle_M$ : modulo-$M)$

In the DFT domain:

$X_d[k] = X[k]\, e^{-j\frac{2\pi}{N}kn_0}$

Let $x[n]$ be real

$X[k] = X^*[N-k]$

$X[k] = X^*[-k]$

Magnitude relation:

$|X[k]| = |X[N-k]|$

$|X[k]| = |X[-k]|$.

Phase relation: $\measuredangle X[k] = -\measuredangle X[N-k]$

**Ex** Circular convolution:

$$x_3[n] = x[n] \; ⑤ \; h[n] = \sum_{m=0}^{N-1} x[m]\, h[\langle n-m\rangle_5]$$

$$x[n] = \begin{cases} 1, & n=0,1 \\ 0, & \text{o.w.} \end{cases} \qquad h[n] = \begin{cases} (0.9)^n, & n=0,1,2,3,4 \\ 0, & \text{o.w.} \end{cases}$$

length of $x[n] = 2$ length $h[n] = 5$

(Use $N = 6 = 5+2-1$ for DFT based convolution.)

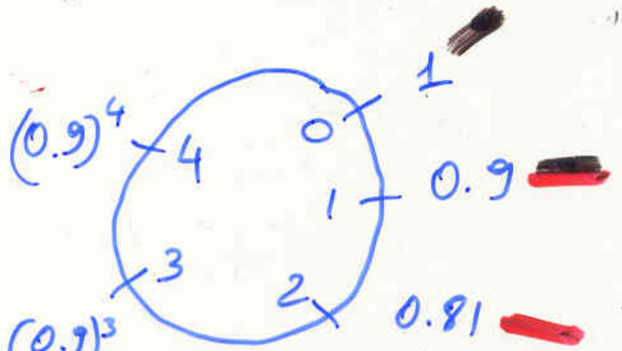
If you use $N=5$ then you have corrupted samples:

$$x_3[0] = \sum_{m=0}^{N=5-1} h[m]\, x[\langle 0-m\rangle_5] = h[0]\, x[\langle 0-0\rangle_5] + h[1]\, \cancel{x[\langle 0-1\rangle_5]}^{\,0} + h[2]\, \cancel{x[\langle -2\rangle_5]} + h[3]\, \cancel{x[\langle -3\rangle_5]} + h[4]\, x[\langle -4\rangle_5]$$

$$= 1 + 0.9^4$$

(Actual conv. value:
$y[0] = h[0]\, x[0] = 1 = \sum_{m=0}^{4} h[m]\, x[0-m]$ )

$x_3[0]$ is corrupted!




$x_3[1] = 1 + 0.9$ , $x_3[2] = 0.9 + 0.81$, $x_3[3] = (0.9)^2 + (0.9)^3$

$x_3[4] = (0.9)^4 + (0.9)^3$

$x_3[0]$ is corrupted by $y[5]$ !

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Another way to evaluate circular convolution

$$x_3[n] = \sum_{\ell=-\infty}^{\infty} y[n + \ell N]$$