Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad \omega \in \mathbb{R}$$

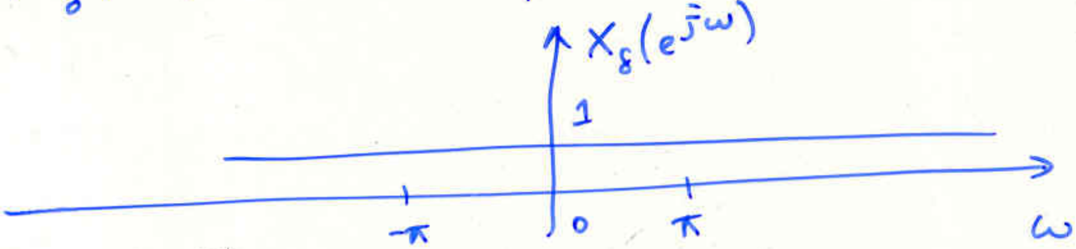
2π periodic

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega, \qquad n = 0, \mp 1, \mp 2, \ldots$$

Ex) Compute the DTFT of $\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{o.w.} \end{cases}$

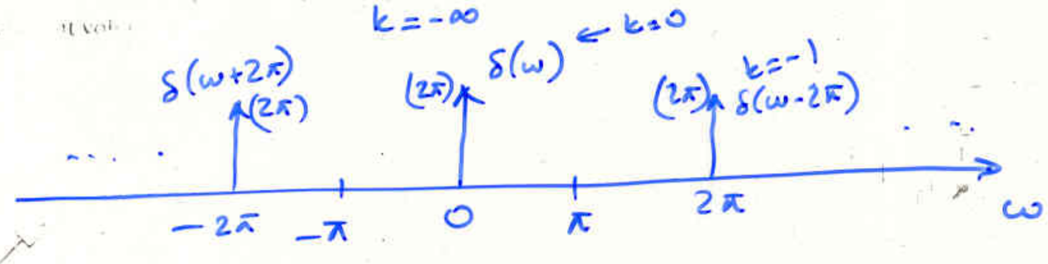
$$X_\delta(e^{j\omega}) = \underbrace{\delta[0]}_{1}\, e^{-j\omega 0} + \overbrace{\delta[1]}^{=0} e^{-j\omega 1} + \cdots + \underbrace{\delta[-1]}_{0} e^{j\omega} + \cdots$$

$$X_\delta(e^{j\omega}) = 1 \quad \text{for all } \omega$$



Ex) Let $x[n] = 1$ for all n, $X(e^{j\omega}) = ?$

Let $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega + 2k\pi)$



$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\, \delta(\omega)\, e^{+j\omega n}\, d\omega = 1, \quad \text{for all } n.$$

Ex) Let $h[n] = 1$ for all n. Is this BIBO stable? No!

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \text{for BIBO stability.}$$

Ex|| $x[n] \doteq \alpha^n u[n]$, Compute $X(e^{j\omega})$.

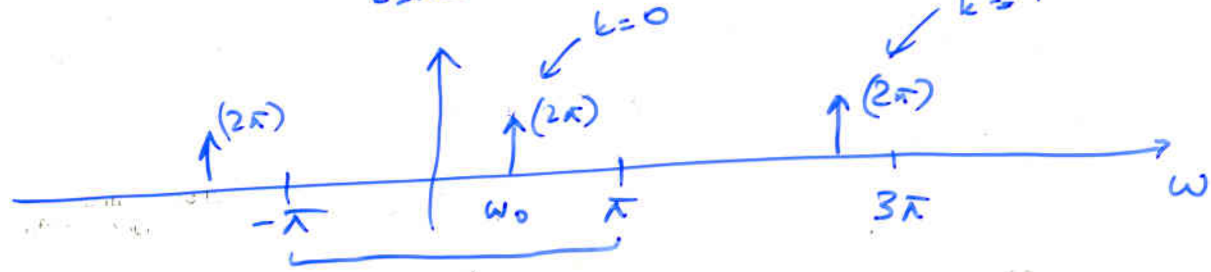
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-(\alpha e^{-j\omega})}, \quad |\alpha e^{-j\omega}|<1$$

$$|\alpha e^{-j\omega}| = |\alpha|\underbrace{|e^{-j\omega}|}_{=1} < 1 \Rightarrow |\alpha| < 1$$

Ex|| $x[n] = e^{j\omega_0 n}$, Compute $X(e^{j\omega})$.

$|e^{j\omega_0}| = 1$ computing the Forward DTFT is tough.

Let $$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0+2\pi k)$$

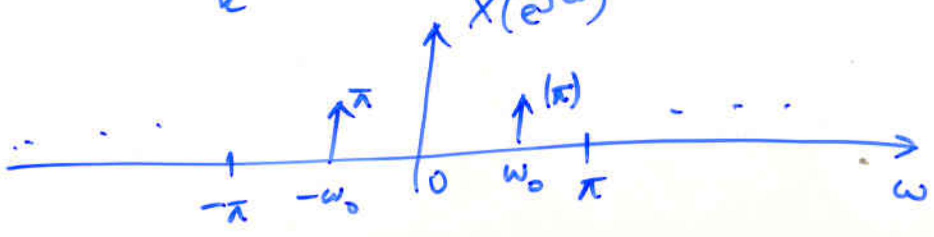


$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\omega-\omega_0)\, e^{j\omega n}\, d\omega = e^{j\omega_0 n}$$

Ex|| $x[n] = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) = \cos\omega_0 n$, Find $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega-\omega_0+2\pi k) + \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega+\omega_0+2\pi k)$$

$$X(e^{j\omega}) = \sum_{k} \pi\,\delta(\omega-\omega_0+2\pi k) + \delta(\omega+\omega_0+2k\pi)$$



Ex|| Let $h[n] = \cos\omega_0 n$, is this BIBO stable? No!

$\sum|\cos\omega_0 n| \not< \infty$

Ex 11. Let $\underset{\text{real}}{x[n]} = x[-n]$, Is $X(e^{j\omega})$ real?

Proof: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = x[0] + x[1]e^{-j\omega 1} + x[2]e^{-j\omega 2} + \dots$
$\qquad\qquad + x[-1]e^{j\omega 1} + x[-2]e^{j\omega 2} + \dots$

$$X(e^{j\omega}) = \underset{\text{real.}}{x[0] + 2x[1]\cos\omega + 2\,x[2]\cos 2\omega + \cdots}$$

Q.E.D.

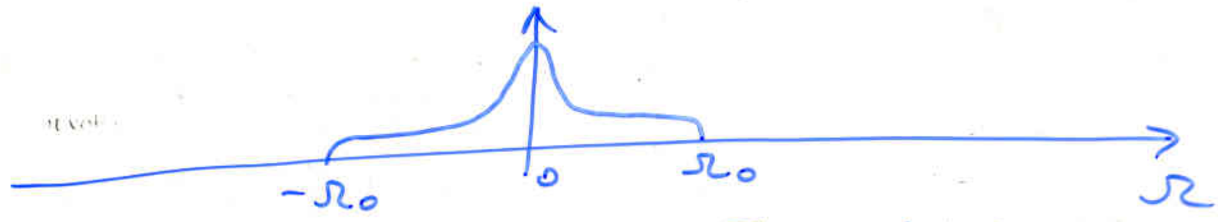
Discrete Fourier Transform: (DFT)

* DFT $\neq$ D.T. F.T.
* DFT is devised to compute (approximately) D.T.F.T and the underlying C.T. F.T.

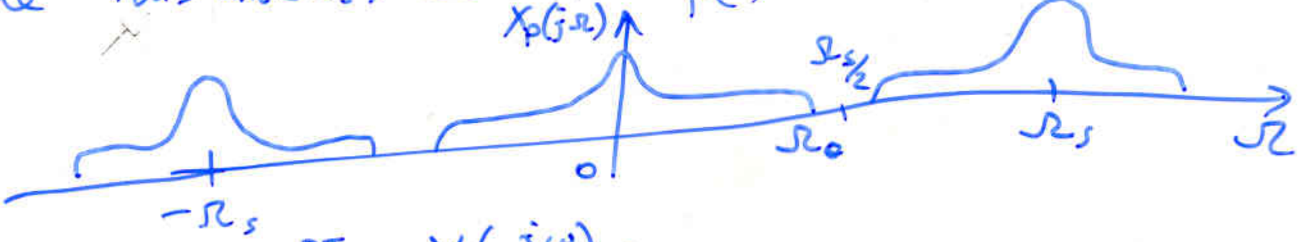
D.F.T. $\quad X[k] \triangleq \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \dots, N-1.$

Let $x_c(t)$ be a bandlimited Cont-time signal.

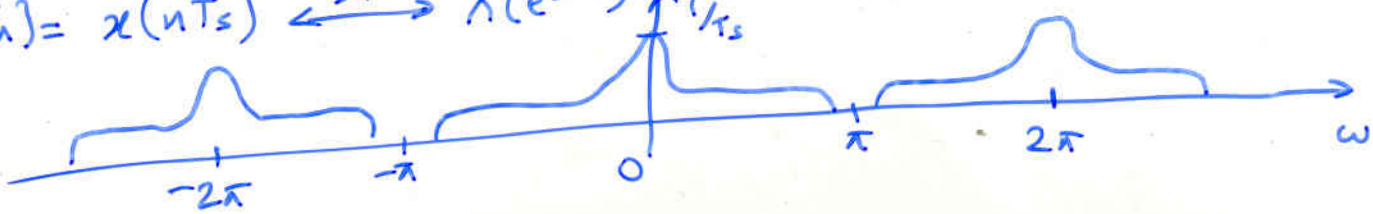
$X_c(j\Omega)$



Sample this with $\Omega_s > 2\Omega_0$: $x_p(t) \xleftrightarrow{\mathcal{F}_{CT}} X_p(j\Omega)$

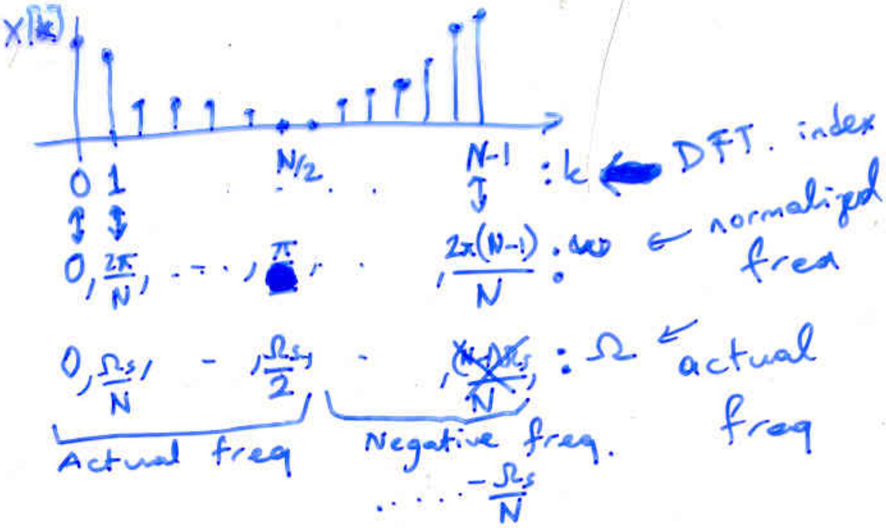


$x[n] = x(nT_s) \xleftrightarrow{DT} X(e^{j\omega})$



DFT_N :

N : even



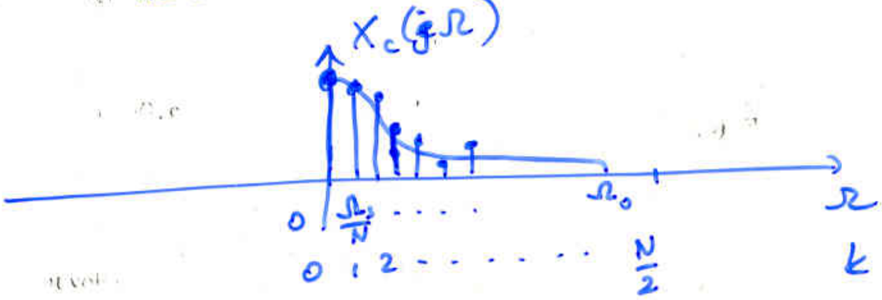
$$X[k] \cong X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$

* The relation is approximate because DFT is a finite sum.

* Actual real frequencies go up to $\frac{\Omega_s}{2}$!

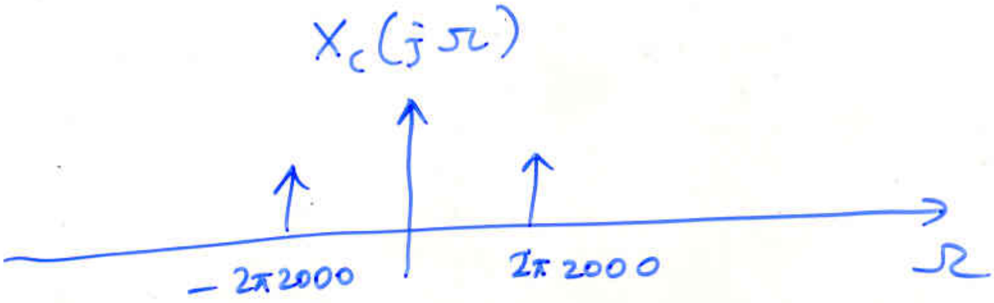
*For an abitrary sequence of numbers we

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \ldots, N-1,$$



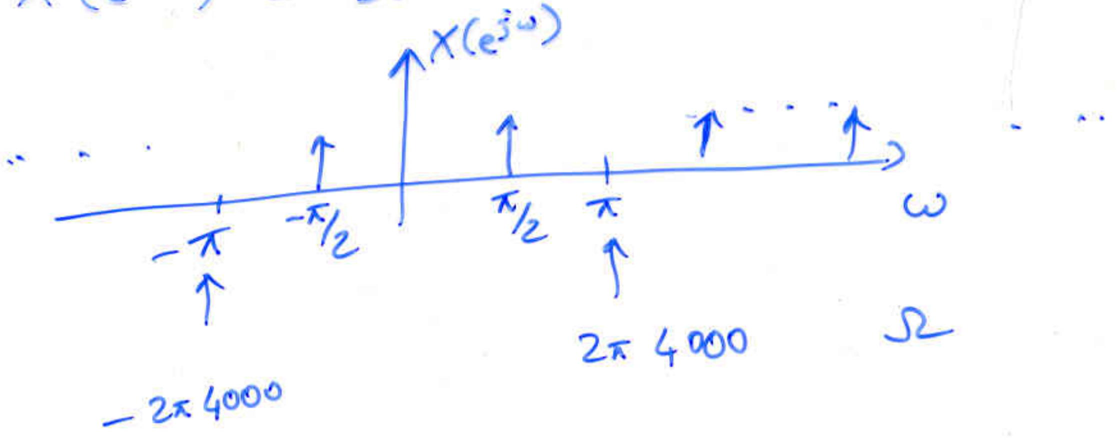
Ex// Let $x_c(t) = \cos 2\pi\, 2000\, t$

Sampling freq: $f_s = 8\,KHz$, $\Omega_s = 2\pi\, 8000$.



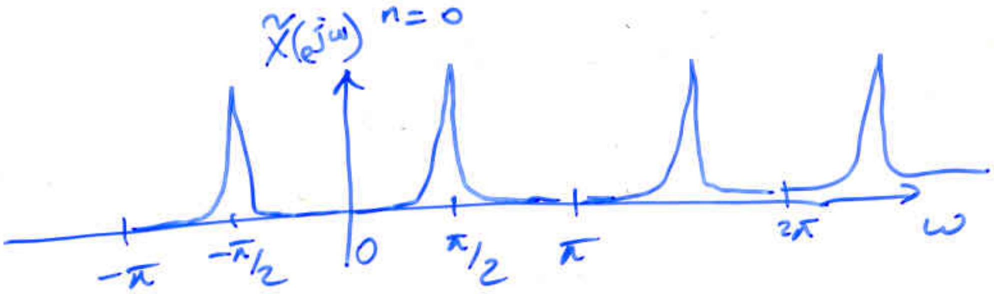
$$x[n] = x_c\left(n\frac{1}{8000}\right), \quad T_s = \frac{1}{8000}, \quad n = 0, \mp 1, \mp 2, \ldots$$

$X(e^{j\omega}) = \text{D.T.F.T}\,(x[n])$



In practice $x[n] = x_c(n\frac{1}{8000})$, $n=0,1,2,\ldots,1023=N-1$

$N=1024$ $\tilde{X}(e^{j\omega}) = \sum_{n=0}^{1023} x[n]e^{-j\omega n} \cong X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

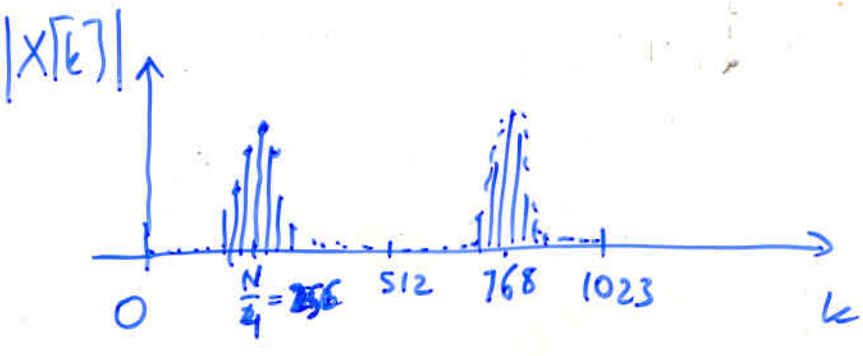


D.F.T. $X[k] = \sum_{n=0}^{1023} x[n]\, e^{-j\frac{2\pi}{N}kn}$, $k=0,1,\ldots,1023$

$X[k] = \tilde{X}(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}$ $k=0,1,\ldots,1023$

DFT and DTFT are equal for finite-extent signals at ω_k

Discrete ω values: $\omega_k = 0, \frac{2\pi}{N}, \frac{2\pi\cdot 2}{N}, \ldots, \frac{2\pi(N-1)}{N} = \frac{2\pi\, 1023}{1024}$



For an arbitrary real signal, $|X(e^{j\omega})| = |X(e^{-j\omega})|$

$X(e^{j\omega}) = X^*(e^{-j\omega})$ so $|X[k]|$ is symmetric wrt $k=\frac{N}{2}$.