Equiripple FIR Filter Design by the FFT Algorithm

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The Fast Fourier Transform (FFT) algorithm has been used in a variety of applications in signal and image processing [1-5]. In this article, a simple procedure for designing Finite-extent Impulse Response (FIR) discrete-time filters using the FFT algorithm is described. The zero-phase (or linear phase) FIR filter design problem is formulated here to alternately satisfy the frequency domain constraints on the magnitude response bounds and time domain constraints on the impulse response support [6-8]. The design scheme is iterative in which each iteration requires two FFT computations. The resultant filter is an equiripple approximation to the desired frequency response.

The main advantage of the FFT-based design method is its implementational simplicity and versatility. Furthermore, the way the algorithm works is intuitive and any additional constraint can be incorporated in the iterations, as long as the convexity property of the overall operations is preserved [6].

In one-dimensional cases, the most widely used equiripple FIR filter design algorithm is the Parks-McClellan algorithm [7]. This algorithm is based on linear programming, and it is computationally efficient. However, it cannot be generalized to higher dimensions. Extension of our design method to higher dimensions is straightforward. In this case two Multi-Dimensional (M-D) FFT computations are needed in each iteration [8].

Zero Phase FIR Filter Specifications and Design Considerations

In this section, the zero-phase FIR filter design problem is described and the notation of the article is introduced.

The term FIR filter refers to a linear shift-invariant system whose input-output relation is represented by a convolution

$$y[n] = \sum_{k \in I} h[k]x[n-k],$$

(1)

where $x[n]$ and $y[n]$ are the input and the output sequences, respectively, $h[n]$ is the impulse response of the filter, and I is the filter support. The FIR filters have only a finite number of nonzero coefficients so that the support I is a bounded region. Usually the filter support, I, is selected as a symmetric region centered at the origin, i.e., $I = \{-N, -N+1, ..., -1, 0, 1, ..., N\}$. The causal FIR filter can be obtained by simply delaying $h[n]$ by N samples.

The problem of designing an FIR filter consists of determining the impulse response sequence, $h[n]$, or its system function, $H(z)$, so that given requirements on the filter response are satisfied. The filter requirements are usually specified in the frequency domain, and only this case is considered here. The frequency response, $H(e^{j\omega})$, corresponding to the impulse response $h[n]$, with a support, I, is expressed as

$$H\left(e^{j\omega}\right) = \sum_{n \in I} h[n]e^{-j\omega n} .$$

(2)

Notice that $H(e^{j\omega})$ is a periodic function with period 2π. This implies that by defining $H(e^{j\omega})$ in the region $\{-\pi < \omega \leq \pi\}$ the frequency response of the filter for all $\omega \in \mathbf{R}$ is determined.

Filter specifications are usually given in terms of the magnitude response, $|H(e^{j\omega})|$. In most applications a two-level magnitude design, where the desired magnitude levels are either 1.0 (in passbands) or 0.0 (in stopbands), is of interest.

Consider the design of a lowpass filter whose specifications are shown in Fig. 1. The magnitude of the lowpass filter ideally takes the value 1.0 in the passband region, $F_p = [-\omega_p, \omega_p]$, and 0.0 in the stopband region, $F_s = [-\pi, -\omega_p] \cup [\omega_p, \pi]$. As magnitude discontinuity is not possible with a finite filter support it is necessary to interpose a transition region, F_t, between F_p and F_s. Also, magnitude bounds $1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p$ in the passband, F_p, and $|H(\omega)| \leq \delta_s$ in the stopband, F_s, are specified, where the parameters δ_p and δ_s are positive real numbers, typically much less than 1.0. Consequently, the lowpass filter is specified in the frequency domain by the regions, F_p, F_s, and the tolerance parameters, δ_p and δ_s. Other widely used filters, bandpass, and highpass filters are specified in a similar manner. The FFT based procedure can easily accommodate arbitrary magnitude specifications as well.

In order to meet given specifications, an adequate filter order (the number of nonzero impulse response samples) needs to be determined. If the specifications are stringent with tight tolerance parameters and small transition regions then the filter support region, I, must be large. In other words, there is a trade-off between the filter support region, I, and the frequency domain specifications. In the general case the filter order is not known *a priori* and may be determined through an iterative process. If the filter order is fixed then the
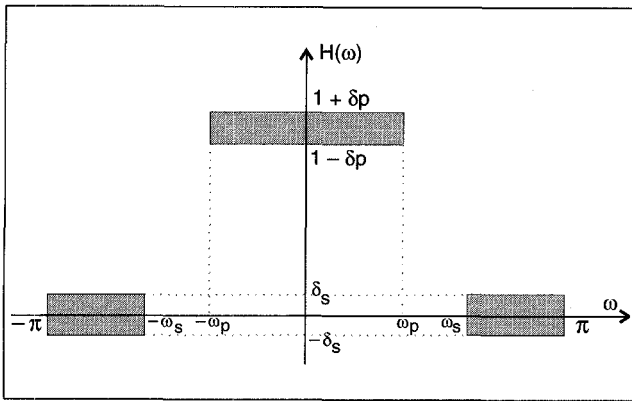
1. Frequency response specifications for the lowpass filter ($1 - \delta_p$ $\leq | H(\omega) | \leq 1 + \delta_p$ for $\omega \in F_p$ and $| H(\omega)| \leq \delta_s$ for $\omega \in F_s$).

tolerance parameters, δ_p and δ_s, must be properly adjusted to meet the design specifications.

Phase linearity or "zero phase" condition is important in many filtering applications [3, 18], and the discussion here is limited to the case of "zero phase" design, with a purely real frequency response. The term "zero phase" is somewhat misleading in the sense that the frequency response may be negative at some frequencies.

In frequency domain the zero-phase or real frequency response condition corresponds to

$$H(e^{j\omega}) = H^*(e^{j\omega}), \tag{3}$$

where $H^*(e^{j\omega})$ denotes the complex conjugate of $H(e^{j\omega})$. The condition (3) is equivalent to

$$h[n] = h^*[-n] \tag{4}$$

in time-domain. Making a common practical assumption that $h[n]$ is real, the above condition reduces to

$$h[n] = h[-n], \tag{5}$$

implying a symmetric filter about the origin.

Iterative Design Method

We now consider the FFT based design procedure, which leads to an equiripple frequency response. In this method we formulate the zero-phase FIR filter design problem in such a way that time and frequency domain constraints on the impulse response are alternately satisfied through an iterative algorithm [6, 9]. The iterative algorithm requires two FFT computations in each iteration.

The frequency response, $H(e^{j\omega})$, of the zero-phase FIR filter is required to be within prescribed upper and lower bounds in its passbands and stopbands. Let us specify bounds on the frequency response $H(e^{j\omega})$ of the FIR filter, $h[n]$, as follows:

$$H_{id}(e^{j\omega}) - E_d(\omega) \leq H(e^{j\omega}) \leq H_{id}(e^{j\omega}) + E_d(\omega) \quad \omega \in F_r, \tag{6}$$

where $H_{id}(e^{j\omega})$ is the ideal filter response, $E_d(\omega)$ is a positive function of ω, which may take different values in different passbands and stopbands, and F_r is the union of passband(s) and stopband(s) of the filter (note that $H(e^{j\omega})$ is real for a zero-phase filter). Usually, $E_d(\omega)$ is chosen constant in a passband or a stopband. For instance,

$$H_{id}(\omega) = \begin{cases} 1, & \text{if } \omega \in F_p \\ 0, & \text{if } \omega \in F_s \end{cases} \tag{7}$$

and

$$E_d(\omega) = \begin{cases} \delta_p, & \text{if } \omega \in F_p \\ \delta_s, & \text{if } \omega \in F_s \end{cases} \tag{8}$$

for the lowpass filter example of the previous section. Inequality (6) is the frequency domain constraint of the iterative filter design method.

In time domain the filter must have a finite-extent support, I, which is a symmetric region around the origin in order to have a zero-phase response (or to achieve phase linearity). The time domain constraint requires that the filter coefficients must be equal to zero outside the region,

$$I = \{n = -N, -N+1, ..., -1, 0, 1, ..., N\} . \tag{9}$$

The iterative method begins with an arbitrary finite-extent, real sequence $h_0[n]$ that is symmetric ($h_0[n] = h_0[-n]$) around the origin. A good candidate for the initial estimate can be obtained from the inverse Fourier Transform of the ideal frequency response $h_{id}[n] = \mathcal{F}^{-1}[H_{id}(\omega)]$ as

$$h_0[n] = \begin{cases} h_{id}[n] & \text{if } n \in I \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

Each iteration of the algorithm consists of making successive imposition of spatial and frequency domain constraints onto the current iterate. The k-th iteration consists of the following steps:

- Compute the Fourier Transform of the k-th iterate $h_k[n]$ on a suitable grid of frequencies by using an FFT algorithm,
- Impose the frequency domain constraint as follows

$$G_k(e^{j\omega}) = \begin{cases} H_{id}(e^{j\omega}) + E_d(\omega) & \text{if } H_k(e^{j\omega}) > H_{id}(e^{j\omega}) + E_d(\omega) \\ H_{id}(e^{j\omega}) - E_d(\omega) & \text{if } H_k(e^{j\omega}) < H_{id}(e^{j\omega}) - E_d(\omega) \\ H_k(e^{j\omega}) & \text{otherwise.} \end{cases} \tag{11}$$

- compute the inverse Fourier Transform of $G_k(e^{j\omega})$
- zero out $g_k[n]$ outside the region I to obtain h_{k+1} as

$$h_{k+1}[n] = \begin{cases} g_k[n], & \text{if } n \in I \\ 0, & \text{otherwise} \end{cases} \tag{12}$$
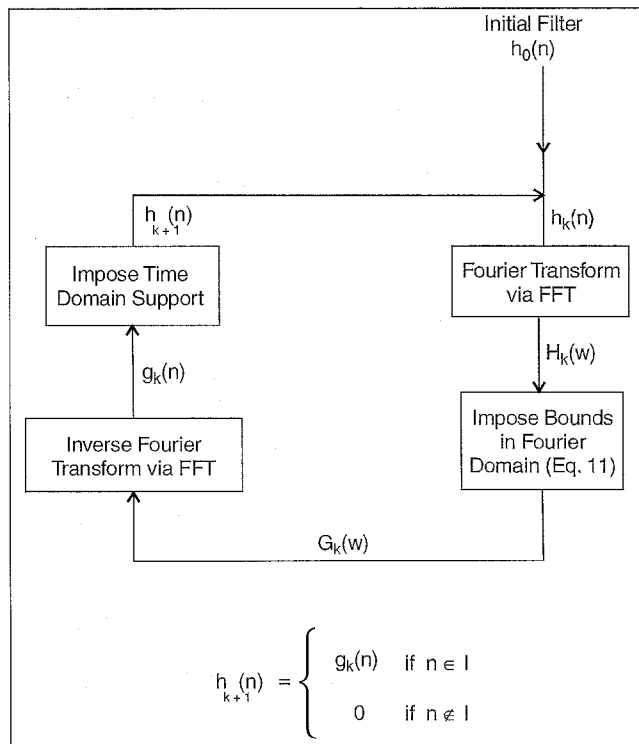
2. *Flow diagram of the iterative filter design algorithm.*

- if the Mean Squared Error between the iterates $h_k[n]$ and $h_{k+1}[n]$ is less than a predefined threshold, then exit.

The flow diagram of this method is shown in Fig. 2. It can be proven that the iterative FIR filter design algorithm is globally convergent. The proof is based on the method of Projections onto Convex Sets (POCS) [10-13]. The constraints (6) and (9) define two convex sets in the set of square summable sequences, $\ell_2 = \left\{ h : \sum_n |h[n]|^2 < \infty \right\}$, and Equations (11) and (12) are the orthogonal projections onto these sets. If the sets intersect then the iterates converge to a member (sequence) in the intersection set. Furthermore, if there is just one sequence satisfying both of the conditions (6) and (9) then this sequence is the equiripple solution of the filter design problem. Consequently, the iterations converge to the equiripple FIR filter.

This method requires the specification of the bounds or equivalently, $E_d(\omega)$, and the filter support, I. If the specifications are too tight then the algorithm does not converge. In such a case one can either progressively enlarge the filter support region, or relax the bounds on the ideal frequency response. In one-dimensional cases there are empirical formulas relating the filter support region and bounds [14]. For lowpass and highpass filters the filter order, $2N + 1$, can be estimated by

$$N = \frac{-20\log_{10}\sqrt{\delta_p \delta_s} - 13}{14.6(\omega_s - \omega_p)/(2\pi)} \tag{13}$$

Other filter order estimates can be found in [7].

Ideally the iterative algorithm should be implemented via the Discrete-Time Fourier Transform. Therefore, the size of the FFT algorithm must be chosen sufficiently large. As a rule of thumb the size of the FFT must be greater than 10 x N for a filter of order $2N + 1$. The passband and stopband edges are very important for the convergence of the algorithm. These edges must be represented accurately on the frequency grid of the FFT algorithm. If the edge frequencies are not on the FFT grid then the frequency values on the grid which render a tighter passband are selected.

Let us now consider an example. We use this example to compare the FTT based design method with the well-known Parks-McClellan algorithm [7].

Example 1: A zero phase half-band filter whose passband and stopband are odd-symmetric around $\omega = \pi/2$ is to be designed. The desired frequency response of the filter is given as follows:

$$H_{id}\left(e^{j\omega}\right) = \begin{cases} 1, & \omega \in \{0 \le \omega \le 0.4\pi\} \\ 0, & \omega \in \{0.6\pi \le \omega \le \pi\} \end{cases} \tag{14}$$

The tolerance parameters are chosen as $\delta_p = \delta_s = 0.05$. In this case a filter of order 11 satisfies the above requirements. The values of the filter coefficients which are obtained after 20 iterations are shown in Table 1A. Notice that the coefficients, $h[2n]$, $n \ne 0$ are negligible compared to $h[0]$. Theoretically these coefficients must be equal to zero due to the odd-symmetric frequency response of the filter. The frequency responses of the iterates after one, two, and 20 iterations are depicted in Fig. 3. The final design is an equiripple approximation to the desired frequency response.

The same filter is also designed using the Parks-McClellan algorithm. The filter coefficients are listed in Table 1B and they are very close to the coefficients of the FFT-based method.

Table 1A. Linear Phase Filter Coefficients Obtained by the FFT-Based Method
$h[0] = 0.50000000 \times 10^0 = h[0]$
$h[1] = 0.31307556 \times 10^0 = h[-1]$
$h[2] = -2.64437737 \times 10^{-16} = h[-2]$
$h[3] = -0.91167637 \times 10^{-1} = h[-3]$
$h[4] = -5.79075137 \times 10^{-7} = h[-4]$
$h[5] = 0.52610448 \times 10^{-1} = h[-5]$

Table 1B. Linear Phase Filter Coefficients Obtained by Parks-McClellan Algorithm
$h[0] = 0.50000000 \times 10^0 = h[0]$
$h[1] = 0.31308165 \times 10^0 = h[-1]$
$h[2] = 0.90987973 \times 10^{-8} = h[-2]$
$h[3] = -0.91150835 \times 10^{-1} = h[-3]$
$h[4] = -0.21467915 \times 10^{-7} = h[-4]$
$h[5] = 0.52676763 \times 10^{-1} = h[-5]$
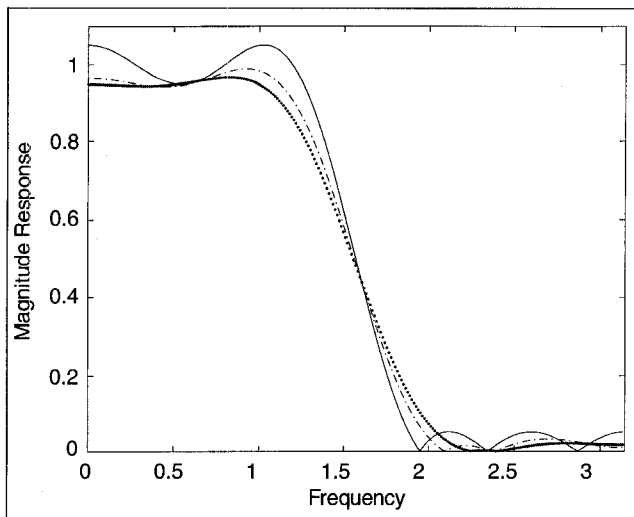
3. Magnitude response of the half-band filter of Example 1 after one (dotted), two (dash-dot) and 20 (solid) iterations.
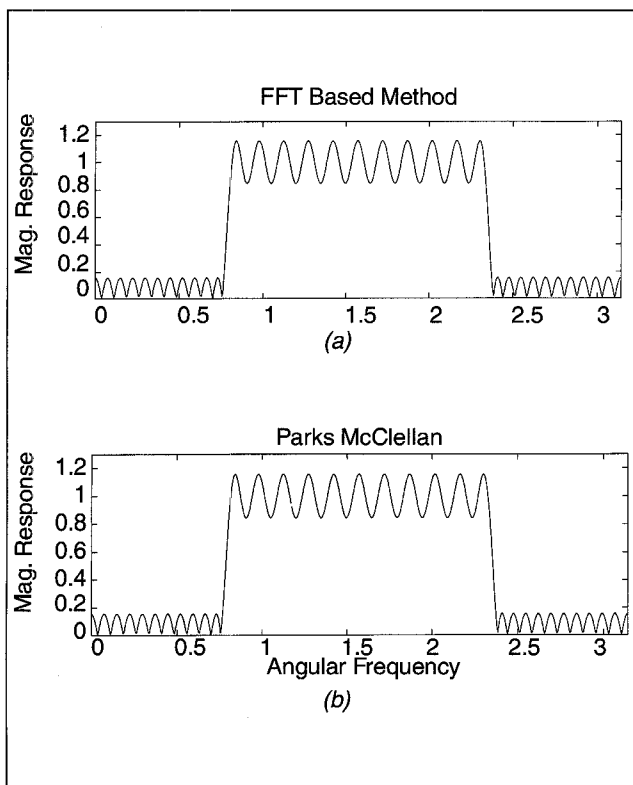
4. Magnitude response of the bandpass filter of order 91 designed by (a) the FFT based algorithm and by (b) Parks McClellan algorithm.

Example 2: A high-order zero-phase bandpass filter is to be designed by using larger filter sizes. The filter constraints are:

$$H_{id}\left(e^{j\omega}\right) = \begin{cases} 0, & \omega \in \{0 \le \omega \le 0.244\pi\} \\ 1, & \omega \in \{0.256\pi \le \omega \le 0.744\pi\} \\ 0, & \omega \in \{0.756\pi \le \omega \le \pi\} \end{cases}$$

The tolerance parameters are chosen as $\delta_p = \delta_s = 0.157$. A filter of order 91 satisfies the given constraints after 130 iterations. Since the transition band is very narrow, the order of the filter is high. The magnitude responses of the filters generated by this algorithm and by Parks-McClellan algorithm are given in Fig. 4.

The design of the above filters does not take more than a few minutes on a workstation.

Multidimensional (M-D) FIR Filter Design

Extension of the FFT based design method to higher dimensions is straightforward. The design of an M-D filter with desired frequency response, $H\left(e^{j\omega_1}, e^{j\omega_2}, ..., e^{j\omega_m}\right)$, can be carried out by defining a multidimensional constraint function $E(\omega_1, \omega_2, ..., \omega_m)$ as in 1-D case. Every iteration of the design method requires two (M-D) FFT computations [8]. The availability of efficient FFT routines makes this procedure advantageous for large-order M-D FIR filter design.

Most of the M-D Equiripple FIR filter design methods are extensions of the 1-D design methods [5], [21]. However, the Parks-McClellan procedure based on the alternation theorem does not find a direct extension to the M-D case. This is because the set of cosine functions used in the 2-D approximation does not satisfy the Haar condition on the domain of interest [15], and the Chebyshev approximation does not have a unique solution. Techniques that employ exchange algorithms [15-17] have been developed for the 2-D case at the expense of increase in computational complexity. Review of these basic design methods for 2-D FIR filters are given in [8].

Example 3: Let us consider the design of a circularly symmetric lowpass filter that leads to an approximately equiripple response. In this case, each iteration requires two 2-D FFT computations.

Maximum allowable deviation is $\delta_p = \delta_s = 0.05$ in both passband and stopband. This means that the 2-D tolerance functions, $E_d(\omega_1, \omega_2) = 0.05$, in the passband and the stopband. The passband and stopband cut-off boundaries have radius of 0.43π and 0.63π, respectively. In the transition band the frequency response is conveniently bounded by the upper bound of the stopband and the lower bound of the passband. The filter support is a square-shaped 17 x 17 region. The frequency response of this filter is shown in Fig. 5.

The shape of the filter support is very important in any M-D filter design method (see e.g. [8, 19, 20]. The support should be chosen to exploit the symmetries in the desired frequency response. For example, diamond-shaped supports show a clear advantage over the commonly assumed rectangular regions in designing 2-D 90° fan filters [22].

Example 4: Let us now consider an example in which we observe the importance of filter support. We design a fan filter whose specifications are shown in Fig. 6a. Maximum allowable deviation is $\delta_p = \delta_s = 0.1$ in both passband and stopband. If one uses a 7 x 7 square-shaped support with 49 points then the design specifications cannot be met. However, a diamond shaped support,
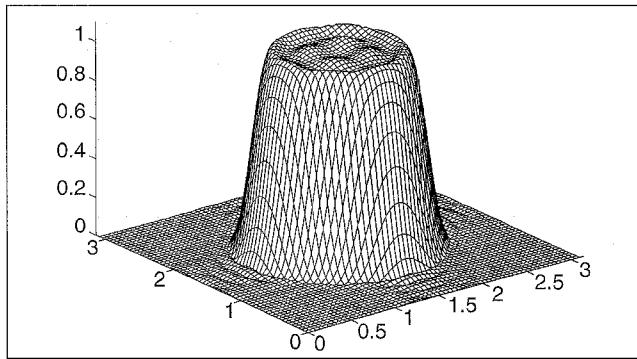
5. Frequency response of the lowpass filter of Example 3.
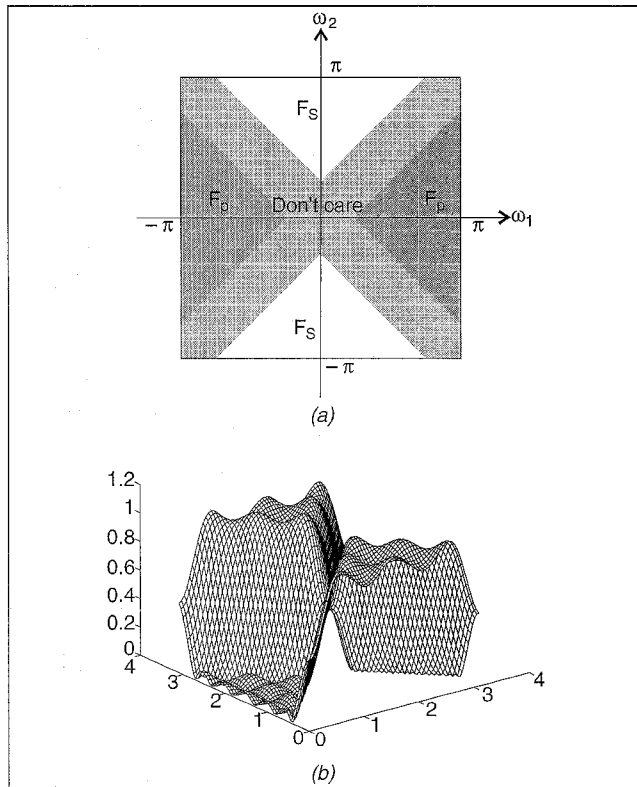
6. (a) Specifications and (b) frequency response of the fan filter designed in Example 4.

$$I_d = \{-5 \leq n_1 + n_2 \leq 5\} \cap \{-5 \leq n_1 - n_2 \leq 5\} , \tag{16}$$

together with the restriction that

$$I_{de} = I_d \cap \{n_1 + n_2 = \text{odd or } n_1 = n_2 = 0\} \tag{17}$$

produces a filter satisfying the bounds. The filter support region, I_{de}, contains 37 points. The resultant frequency response is shown in Fig. 6b.

Conclusions

In this article, an iterative equiripple FIR filter design procedure was described. The iterative algorithm employs the well-known FFT algorithm at each iteration and can be implemented very easily. It can also be generalized to higher dimensions in a straightforward manner.

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