EQUIRIPPLE FIR FILTER DESIGN BY THE FFT ALGORITHM

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Abstract:

Fast Fourier Transform (FFT) algorithm has a wide range of applications in signal and image processing. In this article we describe the design of equiripple Finite Impulse Response (FIR) filters by the FFT algorithm.

1. Introduction

The Fast Fourier Transform (FFT) algorithm has been used in a variety of applications in signal and image processing [1]-[5]. In this article, a simple procedure for designing Finite-extent Impulse Response (FIR) discrete-time filters using the FFT algorithm is described. The zero-phase (or linear phase) FIR filter design problem is formulated here to alternately satisfy the frequency domain constraints on the magnitude response bounds and time domain constraints on the impulse response support [6]-[8]. The design scheme is iterative in which each iteration requires two FFT computations. The resultant filter is an equiripple approximation to the desired frequency response.

Extension of the design method to higher dimensions is straightforward. In this case two Multi-Dimensional (M-D) FFT computations are needed in each iteration [8].

The organization of the article is as follows. A short discussion of characteristics of FIR filters and issues relevant to the design method appears in Section 2. Filter design method is described in Section 3. One or multidimensional FIR filter design examples illustrating the method are also presented.

2. Zero Phase FIR Filter Specifications and Design Considerations

In this section, the zero-phase FIR filter design problem is described and the notation of the article is introduced.

The term FIR filter refers to a linear shift-invariant system whose input-output relation is represented by a convolution

$$y[n] = \sum_{k \in I} h[k]x[n-k], \tag{1}$$

where $x[n]$ and $y[n]$ are the input and the output sequences, respectively, $h[n]$ is the impulse response of the filter, and I is the filter support. The FIR filters have only a finite number of nonzero coefficients so that the support I is a bounded region. Usually the filter support, I, is selected as a symmetric region centered at the origin, i.e., $I = \{-N, -N+1, ..., -1, 0, 1, ..., N\}$. The causal FIR filter can be obtained by simply delaying $h[n]$ by N samples.

The problem of designing an FIR filter consists of determining the impulse response sequence, $h[n]$, or its system function, $H(z)$, so that given requirements on the filter response are satisfied. The filter requirements are usually specified in the frequency domain, and only this case is considered here. The frequency response, $H(e^{j\omega})$, corresponding to the impulse response $h[n]$, with a support, I, is expressed as

$$H(e^{j\omega}) = \sum_{n \in I} h[n]e^{-j\omega n} \quad . \tag{2}$$

Notice that $H(e^{j\omega})$ is a periodic function with period 2π. This implies that by defining $H(e^{j\omega})$ in the region $\{-\pi < \omega \leq \pi\}$ the frequency response of the filter for all $\omega \in \mathbf{R}$ is determined.

Filter specifications are usually given in terms of the magnitude response, $|H(e^{j\omega})|$. In most applications a two-level magnitude design, where the desired magnitude levels are either 1.0 (in passbands) or 0.0 (in stopbands) is of interest.

Consider the design of a lowpass filter whose specifications are shown in Figure .

The magnitude of the lowpass filter ideally takes the value 1.0 in the passband region, $F_p = [-\omega_p, \omega_p]$, and 0.0 in the stopband region, $F_s = [-\pi, -\omega_p] \bigcup [\omega_p, \pi]$. As magnitude discontinuity is not possible with a finite filter support it is necessary to interpose a transition region, F_t, between F_p and F_s. Also, magnitude bounds $1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p$ in the passband, F_p, and $|H(\omega)| \leq \delta_s$ in the stopband, F_s, are specified, where the parameters δ_p and δ_s are positive real numbers, typically much less than 1.0. Consequently, the lowpass filter is specified in the frequency domain by the regions, F_p, F_s, and the tolerance parameters, δ_p and δ_s. Other widely used filters, bandpass and highpass filters are specified in a similar manner. The FFT based procedure can easily accommodate arbitrary magnitude specifications as well.

In order to meet given specifications, an adequate filter order (the number of non-zero impulse response samples) needs to be determined. If the specifications are stringent, with tight tolerance parameters and small transition regions, then the filter support region, I, must be large. In other words, there is a trade-off between the filter support region, I, and the frequency domain specifications. In the general case the filter order is not known a priori and may be determined through an iterative process. If the filter order is fixed then the tolerance parameters, δ_p and δ_s, must be properly adjusted to meet the design specifications.

Phase linearity or "zero phase" condition is important in many filtering applications [3, 10] and the discussion here is limited to the case of "zero phase" design, with a purely real frequency response. The term "zero phase" is somewhat misleading in the sense that the frequency response may be negative at some frequencies. In frequency domain the zero-phase or real frequency response condition corresponds to

$$H(e^{j\omega}) = H^*(e^{j\omega}), \tag{3}$$

where $H^*(e^{j\omega})$ denotes the complex conjugate of $H(e^{j\omega})$. The condition (3) is equivalent to

$$h[n] \;=\; h^*[-n] \tag{4}$$

in time-domain. Making a common practical assumption that $h[n]$ is real, the above condition reduces to

$$h[n] \; = \; h[-n], \tag{5}$$

implying a symmetric filter about the origin.

3. Iterative Design Method

We now consider the FFT based design procedure which leads to an equiripple frequency response. In this method we formulate the zero-phase FIR filter design problem in such a way that time and frequency domain constraints on the impulse response are alternately satisfied through an iterative algorithm [6, 7]. The iterative algorithm requires two FFT computations in each iteration.

The frequency response, $H(e^{j\omega})$, of the zero-phase FIR filter is required to be within prescribed upper and lower bounds in its passbands and stopbands. Let us specify bounds on the frequency response $H(e^{j\omega})$ of the FIR filter, $h[n]$, as follows

$$H_{id}(e^{j\omega}) - E_d(\omega) \leq H(e^{j\omega}) \leq H_{id}(e^{j\omega}) + E_d(\omega) \quad \omega \in F_r, \tag{6}$$

where $H_{id}(e^{j\omega})$ is the ideal filter response, $E_d(\omega)$ is a positive function of ω which may take different values in different passbands and stopbands, and F_r is the union of passband(s) and stopband(s) of the filter (note that $H(e^{j\omega})$ is real for a zero-phase filter). Usually, $E_d(\omega)$ is chosen constant in a passband or a stopband. For instance,

$$H_{id}(\omega) = \begin{cases} 1, & \text{if } \; \omega \in F_p \\ 0, & \text{if } \; \omega \in F_s \end{cases} \tag{7}$$

and

$$E_d(\omega) = \begin{cases} \delta_p, & \text{if } \; \omega \in F_p \\ \delta_s, & \text{if } \; \omega \in F_s \end{cases} \tag{8}$$

for the low-pass filter example of Section 2. Inequality (6) is the frequency domain constraint of the iterative filter design method.

In time domain the filter must have a finite-extent support, I which is symmetric region around the origin in order to have a zero phase response (or to achieve phase linearity). The time domain constraint requires that the filter coefficients must be equal to zero outside the region,

$$I = \{n = -N, -N+1, ..., -1, 0, 1, ..., N\}. \tag{9}$$

The iterative method begins with an arbitrary finite-extent, real sequence $h_0[n]$ that is symmetric ($h_0[n] = h_0[-n]$) around the origin. Each iteration consists of making successive imposition of spatial and frequency domain constraints onto the current iterate. The k-th iteration consists of the following steps:

- Compute the Fourier Transform of the k-th iterate $h_k[n]$ on a suitable grid of frequencies by using an FFT algorithm,
- Impose the frequency domain constraint as follows

$$G_k(e^{j\omega}) = \begin{cases} H_{id}(e^{j\omega}) + E_d(\omega) & \text{if } H_k(e^{j\omega}) > H_{id}(e^{j\omega}) + E_d(\omega), \\ H_{id}(e^{j\omega}) - E_d(\omega) & \text{if } H_k(e^{j\omega}) < H_{id}(e^{j\omega}) - E_d(\omega), \\ H_k(e^{j\omega}) & \text{otherwise.} \end{cases} \tag{10}$$

- compute the inverse Fourier Transform of $G_k(e^{j\omega})$, and
- zero out $g_k[n]$ outside the region I to obtain h_{k+1}.

The flow diagram of this method is shown in Figure . It can be proven that the iterative FIR filter design algorithm is globally convergent, if there exists a solution satisfying both of the conditions (6) and (9).

This method requires the specification of the bounds or equivalently, $E_d(\omega)$, and the filter support, I. If the specifications are too tight then the algorithm does not converge. In such a case one can either progressively enlarge the filter support region, or relax the bounds on the ideal frequency response.

The size of the FFT algorithm must be chosen sufficiently large. The passband and stopband edges are very important for the convergence of the algorithm. These edges must be represented accurately on the frequency grid of the FFT algorithm.

Let us now consider an example. We use this example to compare the FTT based design method with the well-known Parks-McClellan algorithm [9].

Example 1: A zero phase half-band filter whose passband and stopband are odd-symmetric around $\omega = \pi/2$ is to be designed. The desired frequency response of the filter is given as follows

$$H_{id}(e^{j\omega}) = \begin{cases} 1, & \omega \in \{0 \leq \omega \leq 0.4\pi\} \\ 0, & \omega \in \{0.6\pi \leq \omega \leq \pi\} \end{cases} \tag{11}$$

The tolerance parameters are chosen as $\delta_p = \delta_s = 0.05$. In this case a filter of order 11 satisfies the above requirements. The values of the filter coefficients which are obtained after 20 iterations are shown in Table 1-a. Notice that the coefficients, $h[2n], n \neq 0$ are negligible compared to $h[0]$. Theoretically these coefficients must be equal to zero due to the odd-symmetric frequency response of the filter. The frequency response of this filter is depicted in Figure . It is an equiripple approximation to the desired frequency response.

The same filter is designed using the Parks-McClellan algorithm. The filter coefficients are listed in Table 1-b and they are very close to the coefficients of the FFT based method.

4. Multidimensional FIR Filter Design

Extension of the design method to higher dimensions is straightforward. The design of a Multi-Dimensional M-D filter with desired frequency response, $H(e^{j\omega_1}, e^{j\omega_2}, ..., e^{j\omega_m})$, can be carried out by defining a multidimensional constraint function $E(\omega_1, \omega_2, ..., \omega_m)$ as in 1-D case. Every iteration of the design method requires two Multi-Dimensional (M-D) FFT computations [8]. Since there are efficient FFT routines, M-D FIR filters with large orders can be designed by using this procedure.

Example 2: Let us consider the design of a circularly symmetric lowpass filter. Maximum allowable deviation is $\delta_p = \delta_s = 0.05$ in both passband and stopband. The passband and stopband cut-off boundaries have radius of 0.43π and 0.63π, respectively. This means that the 2-D tolerance functions, $E_d(\omega_1, \omega_2) = 0.05$, in the passband and the stopband. In the transition band the frequency response is conveniently bounded by the lower bound of the stopband and the upper bound of the passband. The filter support is a square shaped 17×17 region. The frequency response of this filter is shown in Figure .

The shape of the filter support is very important in any M-D filter design method (see e.g. [8], [11]). The support should be chosen to exploit the symmetries in the desired frequency response. For example, diamond-shaped supports show a clear advantage over the commonly assumed rectangular regions in designing 2-D 90^o fan filters [12].

Example 3: Let us now consider an example in which we observe the importance of filter support. We design a fan filter whose specifications are shown in Figure . Maximum allowable deviation is $\delta_p = \delta_s = 0.1$ in both passband and stopband. If one uses a $7{\times}7$ square-shaped support with 49 points then the design specifications cannot be met. However a diamond shaped support,

$$I_d = \{-5 \leq n_1 + n_2 \leq 5\} \bigcap \{-5 \leq n_1 - n_2 \leq 5\}, \tag{12}$$

together with the restriction that

$$I_{de} = I_d \bigcap \{n_1 + n_2 = \text{odd} \;\; \text{or} \; n_1 = n_2 = 0\} \tag{13}$$

produces a filter satisfying the bounds. The filter support region, I_{de}, contains 37 points. The resultant frequency response is shown in Figure .

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Figure Captions

- Figure 1: Frequency response specifications for the lowpass filter ($1-\delta_p \leq |H(\omega)| \leq 1+\delta_p$ for $\omega \in F_p$ and $|H(\omega)| \leq \delta_s$ for $\omega \in F_s$).
- Figure 2: Flow diagram of the iterative filter design algorithm.
- Figure 3: Magnitude response of the half-band filter of Example 1.
- Figure 4: (a) Frequency response and (b) contour plot of the lowpass filter of Example 2.
- Figure 5:(a) Specifications and (b) frequency response of the fan filter designed in Example 3.

Table Caption

- Table 1: Linear phase filter coefficients obtained by (a) the FFT based method; (b) Parks-McClellan algorithm.