Random Signal:

* Let $x[n]$ be a random variable. (r.v.)

Discrete-time random processes: sequence of r.v.

* We can model the observed signal as a realization of an underlying random process.

* A random process is strict-sense stationary if all the underlying statistical properties are time-independent, i.e., they remain the same for all t.

(i) $f_{x_{n_1}}(x) = f_{x_{n_2}}(x) = f_{x}(x)$ for all n_1, n_2.

(ii) $f_{x_{n_1} x_{n_1+k}}(t_1, t_2) = f_{x_{n_2} x_{n_2+k}}(t_1, t_2)$ for all n_1, n_2, k.

(iii) $f_{x_{n_1} x_{n_1+k} x_{n_1+l}} = f_{x_{n_2} x_{n_2+k} x_{n_2+l}}$ for all k, l, n_1, n_2

⋮

* Wide sense stationary random processes:

(i) $E[x[n]] = \mu$ for all n.

(ii) $E[x[n_1]\, x[n_2]] = E[x[n]\, x[n+k]] = r_x[k]$

for all n, n_1, n_2 and k $|n_2 - n_1| = k$.

W.S.S. $\Longrightarrow$ $r_x[k] = r_x[-k]$.

* $r[k]$ is the auto-correlation seq. of x.

* S.S.S $\Rightarrow$ W.S.S. but W.S.S. $\not\Rightarrow$ S.S.S.

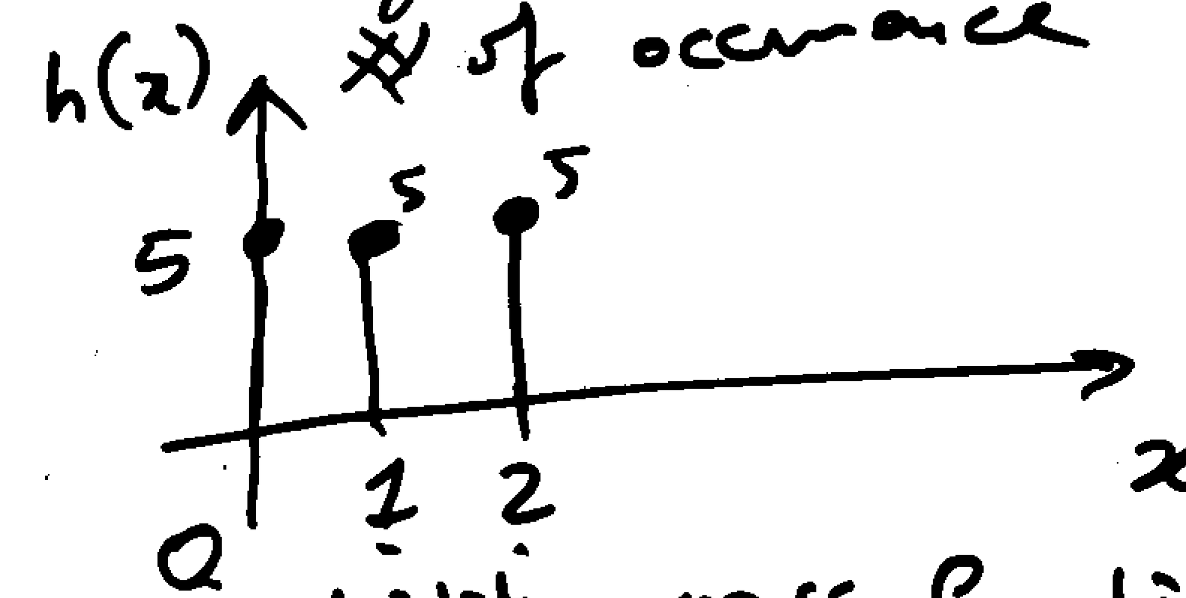
In signal processing we only have a single realization of a random process. in most problems.
So, we assume ergodicity. to replace ensemble averages with time averages.

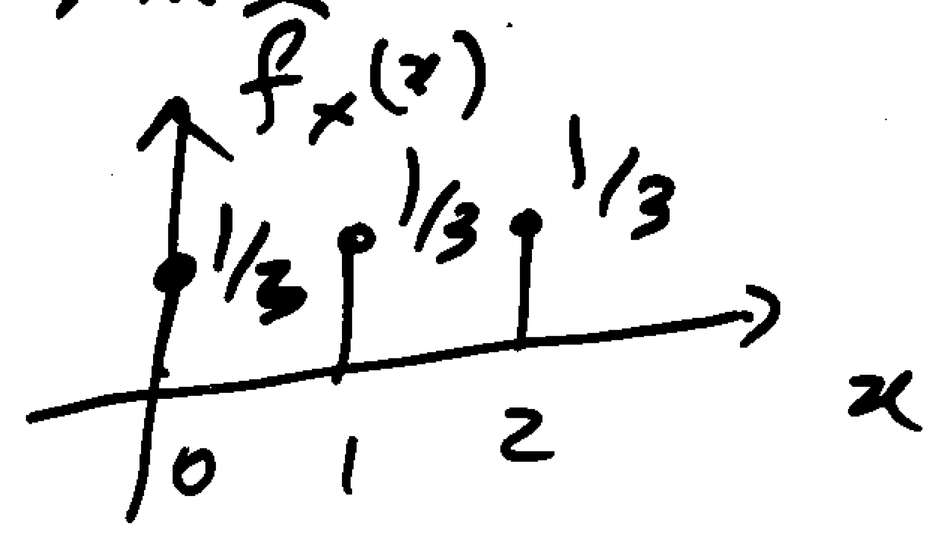
<u>Ex</u> Given the observations:
$x[n] = \{1, 0, 2, 1, 0, 2, 2, 0, 1, 1, 1, 0, 0, 2, 2\}$

Estimate the average value or (mean)

$$\hat{\mu} = \frac{1+0+2+1+\dots+2+2}{15} = 1. = \frac{\sum_{n=0}^{N-1} x[n]}{N}$$

Histogram of the data: An estimate of $f_X(x)$





discrete r.v. probability mass function

$$P_X(x) = \frac{h(x)}{N}$$ where $N = \#$ of observations

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x\left(\frac{1}{3}\delta(x) + \frac{1}{3}\delta(x-1) + \frac{1}{3}\delta(x-2)\right) dx$$

$$= 0 \cdot \int \frac{\delta(x)}{3}\, dx + \int \frac{1\,\delta(x-1)}{3}\, dx + \int \frac{2\,\delta(x-2)}{3}\, dx = \frac{1}{3} + \frac{2}{3} = 1$$

or

$$E(x) = \sum_{x=0}^{2} x\, P_X(x) = 0\,\frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 1$$

Either use the p.m.f. or p.d.f. with impulses.

(r3)

Variance: $E[(X-\overset{1}{\mu})^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$

p.d.f.

$f_X(x) = \frac{1}{3}\delta(x) + \frac{1}{3}\delta(x-1) + \frac{1}{3}\delta(x-2) \equiv \underset{\text{p.m.f.}}{P_X(x)} = \{\underset{\underset{x=0}{\uparrow}}{\tfrac{1}{3}}, \tfrac{1}{3}, \tfrac{1}{3}\}$

$E[(X-1)^2] = \int_{-\infty}^{\infty} (x-1)^2 \frac{\delta(x)}{3}\, dx + \underbrace{\int_{-\infty}^{\infty} (x-1)^2 \delta(x-1)\, dx}_{=1/3} + \int_{-\infty} (x-1)^2 \delta(x-2)\, dx$

$E[(X-1)^2] = \frac{1}{3} + 0 + \int 1^2 \frac{\delta(x-2)}{3}\, dx = \frac{2}{3} = \sigma_x^2$

Using the p.m.f.

$\sigma_x^2 = \sum_{x=0}^{2} (x-1)^2 P_X(x) = (-1)^2 P_X(0) + 0^2 P_X(1) + 1^2 P_X(2)$

$\sigma_x^2 = \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3}$

Estimate the variance from observations:

$\hat{\sigma}_x^2 = \frac{\sum_{n=0}^{14=N-1} (x[n]-\mu)^2}{N} = \frac{0^2 + (-1)^2 + 1^2 + \dots + 1^2 + 1^2}{15} = \frac{10}{15} = \frac{2}{3}$

(Standard deviation $\triangleq \sigma_x$)

Estimate of autocorrelation:

$\hat{r}_x[k] = \frac{1}{N} \sum_{n=0}^{N-|k|-1} x[n]\, x[n+k]$, $\hat{r}_x = \frac{1}{N-|k|} \sum_{n=0}^{N-k-1} x[n]\, x[n+k]$

preferred

Estimates of autocovariance: $c_x[k] = E[(x[n]-\mu)(x[n+k]-\mu)]$ for a w.s.s. r.p.

$\hat{c}_x[k] = \frac{1}{N} \sum_{n=0}^{N-k-1} (x[n]-\mu)^1 (x(n+k)-\mu)^1$ or $\frac{1}{N-k} \sum_{n=0}^{N-k-1} (x[n]-\mu)(x[n+k]-\mu)$

and $c_x[0] = \sigma_x^2$: variance