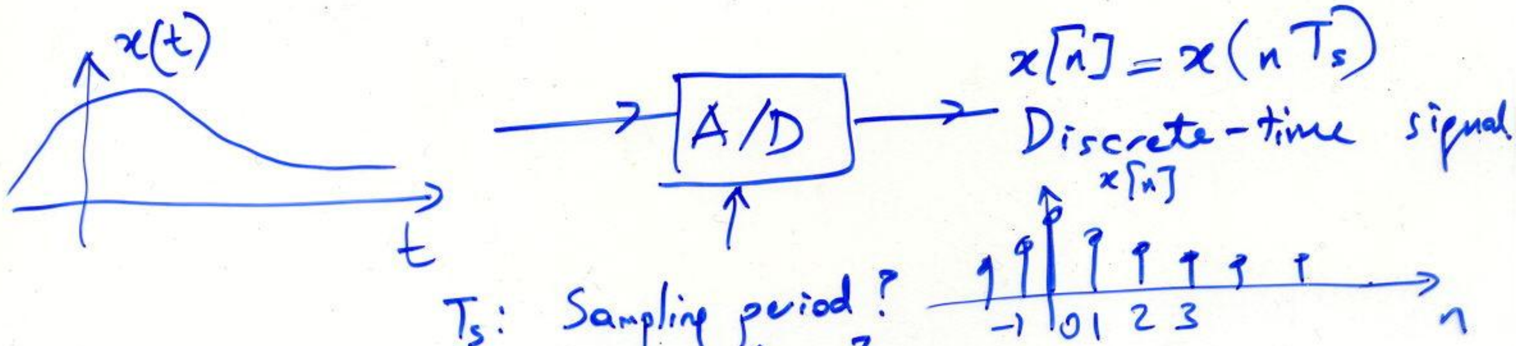
Sampling Theorem

Let $x(t)$ be a C.T. signal.

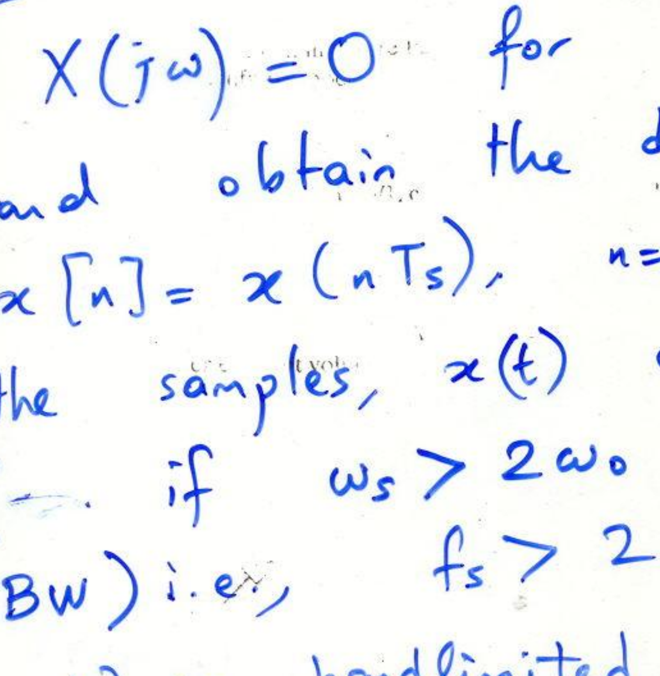


T_s: Sampling period?
f_s: " freq?
$\omega_s = \frac{2\pi}{T_s}$, " ang. freq?

Shannon's Sampling Theorem (Kolmogrov, Whitaker)

Let $x(t)$ be a band-limited signal, i.e., $X(j\omega) = 0$ for $|\omega| > \omega_0$. Sample $x(t)$ and obtain the discrete-time signal $x[n]$: $x[n] = x(nT_s)$, $n = 0, \mp 1, \mp 2, \dots$. Given the samples, $x(t)$ can be perfectly reconstructed if $\omega_s > 2\omega_0$. (sampling freq. > 2 × BW) i.e., $f_s > 2 f_0$ or $\frac{2\pi}{T_s} > 2\omega_0 \dots$

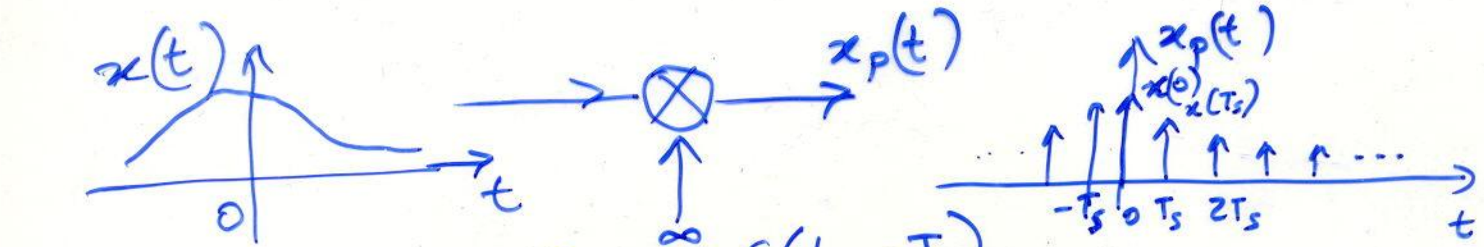
* $x(t)$ is bandlimited:



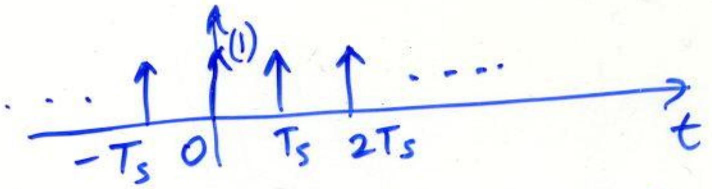
($\omega_s > 2\omega_0$
Nyquist freq.)

* Telephone speech : $f_s = 8\,kHz$. ($f_0 = 4KHz$.)
Music CD : $f_s = 44KHz$.

Proof:



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) = x(t) \cdot p(t).$$

$\equiv$ $x[n] = x(nT_s)$ discrete-time signal

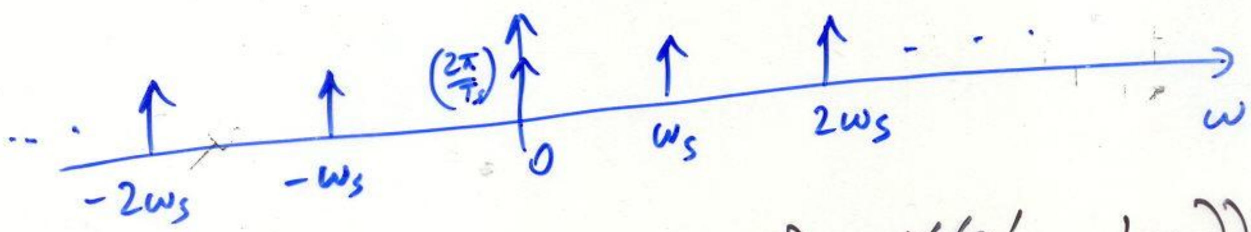
$\equiv$ samples of the cont. time signal.

In Fourie Domain

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

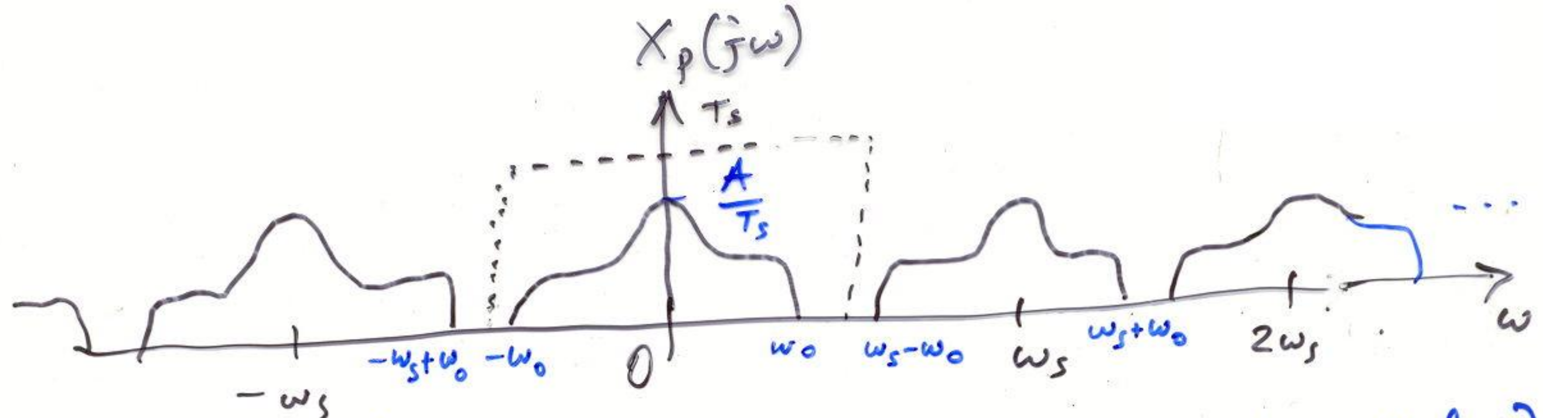
where $P(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T_s} k\right) = \mathcal{F}\{p(t)\}$

$\omega_s = \frac{2\pi}{T_s}.$

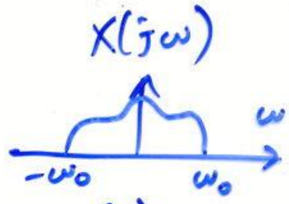


Since $X(j\omega) * \delta(\omega - k\omega_s) = X(j(\omega - k\omega_s))$

$$X_p(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$



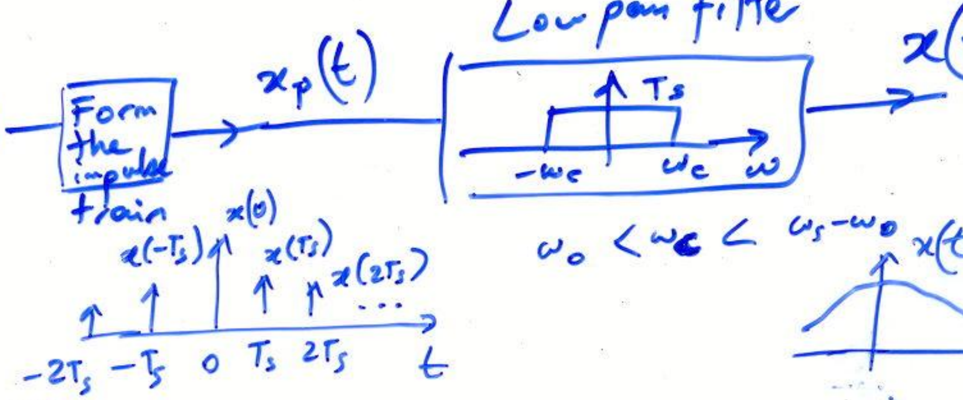
with the assumption $\omega_s - \omega_0 > \omega_0 \Rightarrow \omega_s > 2\omega_0$ (Nyquist freq.)

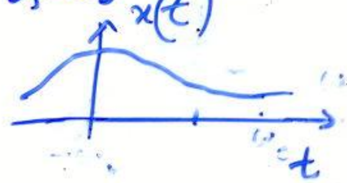


Reconstruction:

$x[n] = x(nT_s)$

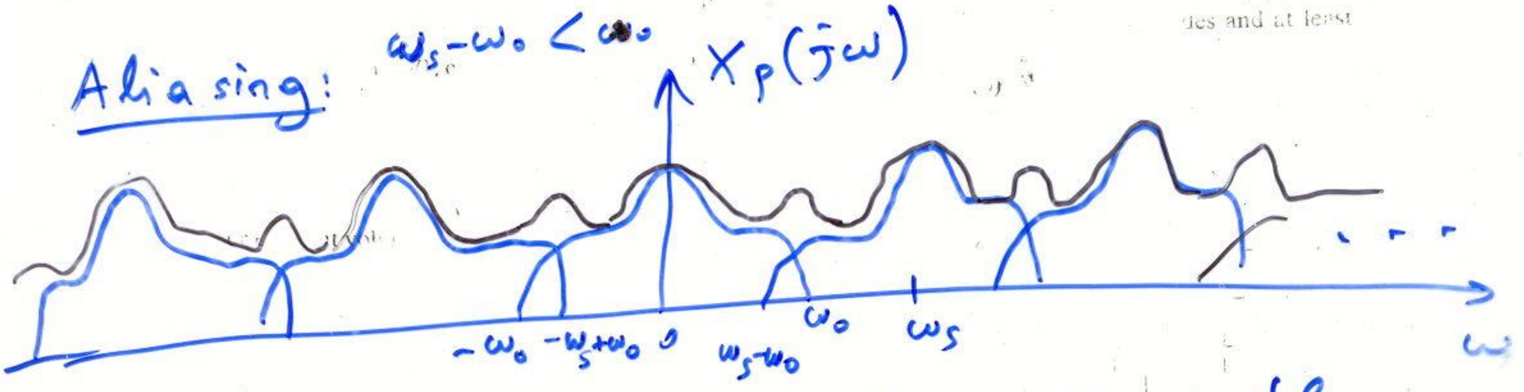
Seq. of numbers.





Aliasing:

$\omega_s - \omega_0 < \omega_0$



There is overlap and it is impossible to reconstruct $x(t)$ from $x_p(t)$ or equivalently from the samples $x[n] = x(nT_s)$, $n = 0, \mp 1, \mp 2, \ldots$

* Time domain example: stage-coach in Cowboy movies.

Interpolation Formula:

$$x_p(t) \longrightarrow \boxed{H_{LP}(j\omega)} \longrightarrow x(t)$$

impulse train obtained from the signal samples.

original

$$x(t) = x_p(t) * h_{\ell p}(t)$$

where $\quad h_{\ell p}(t) = T\,\frac{\omega_c}{\pi}\,\text{sinc}\left(\frac{\omega_c t}{\pi}\right)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

$$\left(\delta(t-t_0) * h(t) = h(t-t_0)\right)$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\left(\underbrace{h_{\ell p}(t) * \delta(t-nT)}_{h_{\ell p}(t-nT)}\right)$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,T\,\frac{\omega_c}{\pi}\,\text{sinc}\left(\frac{\omega_c(t-nT)}{\pi}\right)$$

where $\quad T$: sampling period

ω_c : cut-off freq. of the lowpass filter.

(usually $\omega_c = \frac{\omega_s}{2} = \frac{2\pi}{T\cdot 2} = \frac{\pi}{T}$)

RHS: $\quad x(0), x(\mp T), x(\mp 2T), \ldots$

LHS: $\quad x(t)$ cont.-time signal

The signal value at an arbitrary $t = t_0$:

$$x(t_0) = \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}\left(\frac{\omega_c(t_0-nT)}{\pi}\right), \quad \omega_c = \frac{\omega_s}{2}$$

Nyquist Frequency:

$$\omega_s > 2\omega_M$$

Ex

$$y[n] = \frac{1}{4}x[n+1] + \frac{1}{2}x[n] + \frac{1}{4}x[n-1]$$

has an anticausal impulse response:

$$h[n] = \left\{ \frac{1}{4},\ \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}},\ \frac{1}{4} \right\}$$

In an FIR filter, filter coefficients determine the impulse response.

Ex

$$y[n] = \text{median}\left\{ x[n+1], x[n], x[n-1] \right\}$$

is a nonlinear system.