White-Noise : $u[n]$

$$r_u[0] = \sigma_u^2 \neq 0 .$$

$$r_u[k] = 0 \quad , \text{ for } \quad k = \mp 1, \mp 2, \dots$$

There is no correlation between the samples of $u[n]$.

i.i.d. $\equiv$ independent and identically distributed.

Each sample has the same p.d.f. but they are all independent. $f_{u_j u_k}(t_1, t_2) = f_{u_j}(t_1) f_{u_k}(t_2)$ for $j \neq k$.

independence $\Longrightarrow$ uncorrelatedness

$\not\Longleftarrow$

$$f_{u_j}(t_1) f_{u_k}(t_2) = f_{u_j u_k}(t_1, t_1)$$

$$E[u[j]\, u[k]] \;(= 0 \text{ (for zero mean r.p.)})$$

$$= E[u[j]]\, E[u[k]]$$

for an arbitrary r.p.

Theorem: L.T.I systems don't disturb w.s.s.

L.T.I System

$$u[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

w.s.s.: white noise

w.s.s.

$y[n]$ is also w.s.s. . This is also valid for any w.s.s. input $x[n]$.

Def'n// Given a w.s.s. r.p. $x[n]$ with auto-correlation $r_x[k]$. The spectrum of $x[n]$ is defined as the D.T.F.T. of $r_x[k]$:

$$S_x(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} r_x[k]\, e^{-j\omega k}$$

* Since $r_x[k] = r_x[-k]$ the spectrum $S_x(e^{j\omega})$ is real! $\Rightarrow$ No phase term!

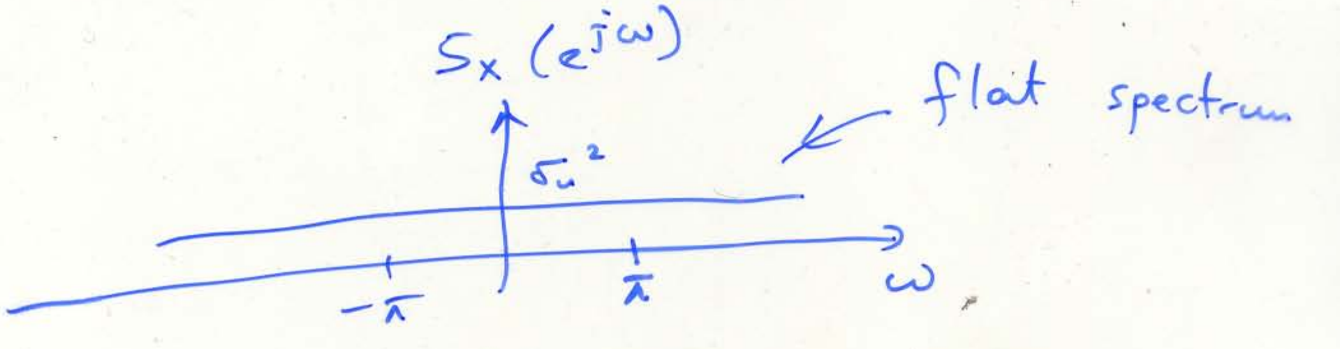
* $S_x(e^{j\omega})$ is 2π periodic

Ex// Spectrum of white-noise:

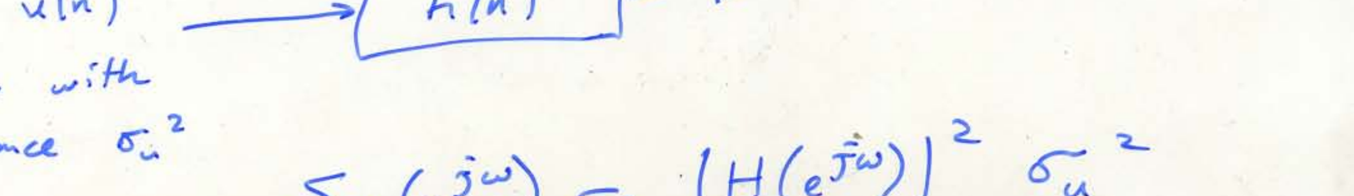
$$r_u[k] = \begin{cases} \sigma_u^2, & k=0 \\ 0, & k \neq 0 \end{cases}$$

$$S_x(e^{j\omega}) = r_x[0] \cdot e^{-j\omega \cdot 0} + 0 \cdot e^{-j\omega 1} + 0 \cdot e^{j\omega 1} + \dots$$

$$= \sigma_u^2 \quad \text{for all } \omega$$



Theorem:



white with variance σ_u^2

$$S_y(e^{j\omega}) = |H(e^{j\omega})|^2 \sigma_u^2$$

In general $\quad S_y(e^{j\omega}) = |H(e^{j\omega})|^2 S_x(e^{j\omega})$, x is w.s.s. input.

Ex// Let $y[n] = \frac{1}{2} x[n] + \frac{1}{2} x[n-1]$

FIR system or Moving Average (MA) system.

Calculate $r_y[n]$. and $S_y(e^{j\omega})$ given that the input is white, with variance σ^2. (zero-mean)

Autocorrelation sequence:

$$r_y[0] = E\left[\left(\frac{1}{2} x[n] + \frac{1}{2} x[n-1]\right)\left(\frac{1}{2} x[n] + \frac{1}{2} x[n-1]\right)\right]$$

$$= E\left[\frac{1}{4} x[n]^2 + 2\frac{1}{4} x[n] x[n-1] + \frac{1}{4} x[n-1]^2\right]$$

$$= \frac{1}{4} E\left[x^2[n]\right] + \frac{1}{2} E\left[\cancel{x[n] x[n-1]}\right]^{0} + \frac{1}{4} E\left[x^2[n-1]\right]$$

$$r_y[0] = \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 = \frac{1}{2}\sigma^2$$

$$r_y[1] = E\left[y[n]\, y[n-1]\right] = E\left[y[1]\, y[0]\right] = \dots$$

$$= E\left[\frac{1}{2}\left(x[1] + x[0]\right)\left(\frac{1}{2} x[0] + \frac{1}{2} x[-1]\right)\right]$$

$$= \frac{1}{4} E\left[x[1] x[0] + x[0]^2 + x[-1] x[1] + x[-1] x[0]\right]$$

$$= \frac{1}{4}\left(E\left[\cancel{x[1] x[0]}\right]^{0} + E\left[x^2[0]\right] + 0 + 0\right)$$

$$r_y[1] = \frac{1}{4}\sigma^2 = r_y[-1]$$ (from symmetry of autocorrelation).

$$r_y[2] = E\left[y[n]\, y[n-2]\right] = E\left[\frac{1}{2}\left(x[n] + x[n-1]\right)\frac{1}{2}\left(x[n-2] + x[n-3]\right)\right]$$

$$= 0 = r_y[-2]$$

$$r_y[3] = r_y[-3] = 0 \dots$$

<u>Ex#</u> Let $y[n] = \alpha_1 y[n-1] + u[n]$ (all-pole system) IIR, recursive,

where $u[n]$ is a zero-mean, white, w.s.s. . Determine the first order predictor $\hat{y}[n] = a_1 y[n-1]$.

First calculate the autocorrelation sequence of y.

$$r_y[0] = E[y[n] \cdot y[n]] = E[(\alpha_1 y[n-1] + u[n])(\alpha_1 y[n-1] + u[n])]$$

$$= \alpha_1^2 E[y[n-1]\, y[n-1]] + 2\alpha_1 E[\cancel{y[n-1] u[n]}]^{0} + E[u^2[n]]$$

$$r_y[0] = \alpha_1^2 r_y[0] + 0 + \sigma_u^2$$

$$r_y[0] = \frac{\sigma_u^2}{1-\alpha_1^2}$$. σ_u^2 is the variance of $u[n]$

$$r_y[1] = E[y[n] \cdot y[n-1]] = E[(\alpha_1 y[n-1] + u[n])\, y[n-1]]$$

$$r_y[1] = \alpha_1 E[y^2[n-1]] + E[\cancel{u[n]\, y[n-1]}]^{0}$$

$$r_y[1] = \alpha_1 r_y[0] = \alpha_1 \frac{\sigma_u^2}{1-\alpha_1^2}$$

$$r_y[2] = E[y[n]\, y[n-2]] = E[(\alpha_1 y[n-1] + u[n])\, y[n-2]]$$

$$r_y[2] = \alpha_1 E[y[n-1]\, y[n-2]] + 0 = r_y[1]$$

$$r_y[2] = \alpha_1 r_y[1] = \alpha_1^2 \cdot \sigma_u^2 / (1-\alpha_1)^2 \quad \ldots$$

A.C.N.E becomes: (1 by 1 matrix)

$$r_y[1] = r_y[0] \cdot a_1$$

$$a_1 = \frac{r_y[1]}{r_y[0]} = \alpha_1$$, Predictor: $\hat{y}[n] = \alpha_1 y[n-1]$