Name: **ID:** **Section:**

Score: / 100

EEE-424 Digital Signal Processing: Final-Term Exam 2009/2010

Duration: 2 hours
Instructions: No calculators, book or notes allowed. SHOW YOUR WORK! No credit for results without explanations or steps!!

Q.1

Given $x[n] = \{1, 4, 1, 4, 1, 1\}$ starting at index zero.

Q.1a (3 pts) Calculate the estimated mean $\hat{m}_x$ and estimated variance $\hat{\sigma}_x^2$ of $x[n]$.

Solution : $m_x = 2,\ \sigma_x^2 = 2.$

Q.1b (11 pts) Propose a third order anticausal FIR filter $h[n]$ with $h[0] = 1$ so that the output signal $y[n], n = 1, 2, 3, 4$ has smaller variance than the input signal $x[n]$. Calculate (approximately) the variance $\hat{\sigma}_y^2$ of $y[n], n = 1, 2, 3, 4$. Specify the type of your proposal filter $h[n]$.

Solution : One possible sol: $h[-1] = h[1] = 0.5 \Rightarrow (\hat{\sigma}_y^2 \approx 1.3/4) \leq \sigma_x^2$. Lowpass.

Q.1c (11 pts) Propose a third order anticausal FIR filter $g[n]$ with $g[0] = 1$ so that the output signal $z[n] = g[n] * x[n], n = 1, 2, 3, 4$ is zero-mean. Specify the type of your proposal filter $g[n]$.

Solution : Proposal: $g[n] = a\delta[n-1] + \delta[n] + a\delta[n+1] \Rightarrow \sum_{n=1}^{4} z[n] = 17a + 10 \stackrel{!}{=} 0 \Rightarrow a = -10/17$. Highpass.

Q.2

Design a Lowpass Butterworth filter with $w_c = \pi/2$ and filter order $N = 1$.

Q.2a (5 pts) Determine the analog prototype filter $H_a(s)$.

Solution : $\Omega_c = \frac{2}{T}\tan(\pi/4) = \frac{2}{T} \Rightarrow |H_a(s)|^2 = \frac{1}{1+\left(\frac{s}{j2/T}\right)^2}$.

Q.2b (5 pts) Determine all poles of $H_a(s)$.

Solution : $s_1 = 2/T,\ s_2 = -2/T$

Q.2c (10 pts) Apply bilinear transform to determine a stable digital filter $H(z)$.

Solution : Use s_2 in the left half plane: $H_a(s) = \frac{-s_2}{s-s_2} \Rightarrow H(z) = \frac{1}{2}(1 + z^{-1})$.

Q.2d (5 pts) Determine the Region Of Convergence (ROC) of the digital filter.

Solution : ROC=$\mathbb{C}/\{0\}$.

Q.3

Given $x_1[n] = \{-1, -2, 0, 1, -1, 3\}$ and $x_2[n] = \{1, 1, -3, 2, 1, -3, 1\}$, both W.S.S. and starting at index zero.

Q.3a (12 pts) Which signal is more likely white noise? Explain your answer by computing a suitable statistical measure for both signals and comparing it.

Solution : $\hat{r}_{x_1} = \{16/6, -1/3, 1/6, 1/6, ...\}$, $\hat{r}_{x_2} = \{26/7, -12/7, -9/7, 2, ...\}$. x_1 is more likely white noise.

Q.3b (5 pts) Consider $y_1[n] = h[n] * x_1[n]$, where $h[n]$ is an FIR filter. Is $y_1[n]$ W.S.S.? Explain your answer.

Solution : FIR is an LTI system, which is linear and stable. Mean and variance are preserved over time.

Q.3c (8 pts) Let $w[n]$ be a W.S.S. zero mean white noise signal with variance 1. Compute its spectrum $S_w(e^{j\omega})$. Comment on your result.

Solution : $r_w[n] = \delta[n] \Rightarrow S_w(e^{j\omega}) = 1 \ \forall\ \omega$.

Q.4

Given the sequence of W.S.S. random numbers $x[n] = \{1, 1, 2, 3, 5\}$ starting at index zero.

Q.4a (15 pts) A second order linear MMSE predictor $\hat{x}[n] = a_1 x[n-1] + a_2 x[n-2]$ shall be determined by following method: Find a_1, a_2 such that $g(a_1, a_2) = \sum_{n=2}^{4} (x[n] - \hat{x}[n])^2$ is minimized. Do NOT use the ACNE (no autocorrelations!). Hint: Use derivatives $\frac{dg(a_1,a_2)}{da_1}$ and $\frac{dg(a_1,a_2)}{da_2}$.

Solution : $a_1 = a_2 = 1$.

Q.4b (3 pts) Compute the predictions $\hat{x}[2], \hat{x}[3], \hat{x}[4]$ and $\hat{x}[5]$ using the predictor from [Q.4a].

Solution : $\hat{x}[2] = 2, \hat{x}[3] = 3, \hat{x}[4] = 5, \hat{x}[5] = 8$.

Q.4c (7 pts) Consider $y[n] = \frac{\sin(n)}{n}$, $n \geq 0$. Is $y[n]$ W.S.S.? Explain your answer!

Solution : This is a deterministic signal. One can compute the properties mean and autocorrelation though. Mean is zero. For the autocorrelation, we can write

$$|r_y[k]| = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{\sin(n)}{n} \frac{\sin(n+k)}{n+k} \right| \tag{1}$$

$$\leq \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{\sin^2(n)}{n^2} \right| \tag{2}$$

$$\leq \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{n^2} \right| = 0 \ \forall k \tag{3}$$

These properties remain constant, so this signal, though being deterministic, has a mean and autocorrelation and they are constant. It satisfies W.S.S properties.

Formulas

- Estimated mean:

$$\hat{m}_x = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

- Estimated variance:

$$\hat{\sigma}_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x[n]-\hat{m}_x)^2$$

- Estimated autocorrelation (for zero mean seq.):

$$\hat{r}_x[k] = \frac{1}{N}\sum_{n=0}^{N-k-1} x[n]x[n+k],\ \ k \geq 0$$

- Spectrum of random signal $x[n]$:

$$S_x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} r_x[n]e^{-j\omega n}$$

- Butterworth filter:

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}},\ \ |H_a(s)|^2 = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

- Poles of Butterworth filter:

$$s_k = \Omega_c e^{j\frac{\pi}{2}(1+\frac{2k+1}{N})},\ \ k = 1, ..., 2N$$

- Bilinear transform:

$$\Omega = \frac{2}{T}\tan(\omega/2),\ \ H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$