EE 424 – Digital Signal Processing

Midterm

Fall 2010

Duration: 100 minutes

Name: ....

Student ID: ....

Section: ....

Question-1 (25 pts)	
Question-2 (25 pts)	
Question-3 (25 pts)	
Question-4 (25 pts)	
TOTAL (100 pts)	

$$X[k]=\sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \quad 0 \leq k \leq N-1$$

Q1) Let $x[n]=\{1,\ 2,\ 3\}$, $\underset{n=0}{\uparrow}$

a) Calculate the 4-point DFT of $x[n]$. $\Rightarrow x[n]=\{1, 2, 3, 0\}$

b) Calculate the 64-point DFT of $x[n]$. $\Rightarrow x[n]=\{1, 2, 3, 0 \dots 0\}$

c) Calculate the 64-point DFT of $h[n]=\{1,\ 2,\ 1\}$, $\underset{n=0}{\uparrow}$ $\Rightarrow x[n]=\{2, 1, 0, \dots 0, 1\}$

a) $$X[k]=\sum_{n=0}^{3} x[n] e^{-j2\pi kn/4}$$

$$= 1.e^0 + 2.e^{-j2\pi k/4} + 3e^{-j4\pi k/4}$$

$$\Rightarrow X[0] = 1+2+3 = 6$$

$$X[1] = +1-2j-3 = -2-2j$$

$$X[2] = 1-2+3 = 2$$

$$X[3] = 1+2j-3 = -2+2j$$

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b) $$X[k]=\sum_{n=0}^{63} x[n].e^{-j2\pi kn/64}$$

$$= 1.e^0 + 2.e^{-j2\pi k/64} + 3.e^{-j4\pi k/64} + \underbrace{0.e^{-j6\pi k/64} \dots + 0.e^{-j126\pi k/64}}_{0}$$

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c) $$X[k]=\sum_{n=0}^{63} x[n].e^{-j2\pi kn/64}$$

$$= 2.e^0 + 1.e^{-j2\pi k/64} + \underbrace{0.e^{-j6\pi k/64} \dots}_{0} + 1e^{-j126\pi k/64}$$

$$= 2.e^0 + \underbrace{1.e^{-j2\pi k/64} + 1.e^{-j126\pi k/64}}$$

$$= 2.e^0 + e^{-j2\pi k/64} + e^{j2\pi k/64}$$

$$= 2+2\cos(2\pi k/64)$$

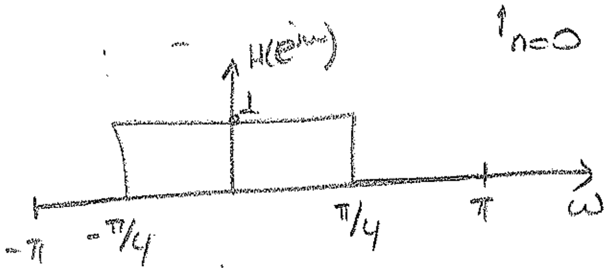
Q2)

a) Design a 7th order low-pass FIR filter with cut-off $\omega_c = \frac{\pi}{4}$ using the rectangular window.

b) Explain why the filter that you designed in part (a) is not a good filter.

c) Design the same filter using a triangular window.

a) $\hat{h}[n] = \{h_{-3}, h_{-2}, h_{-1}, \underset{\uparrow n=0}{h_0}, h_1, h_2, h_3\}$ & $w[n] = \{1, 1, 1, \underset{\uparrow n=0}{1}, 1, 1, 1\}$



$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) \cdot e^{j\omega n}\, d\omega = \frac{1}{2\pi} \cdot \int_{-\pi/4}^{\pi/4} e^{j\omega n}\, d\omega = \frac{1}{2\pi} \cdot \frac{1}{jn} \cdot e^{j\omega n}\Big]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi n} \cdot \left[\frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j}\right] = \boxed{\frac{\sin(\pi/4 \cdot n)}{\pi n} = h[n]}$$

$\Rightarrow \hat{h}[n] = h[n] \cdot w[n]$, $n = -3, -2, -1, 0, 1, 2, 3$

$$\hat{h}[n] = \left\{ \frac{\sin\frac{-3\pi}{4}}{-3\pi}, \frac{\sin\frac{-2\pi}{4}}{-2\pi}, \frac{\sin\frac{-\pi}{4}}{-\pi}, \frac{\sin 0}{0}, \frac{\sin \pi/4}{\pi}, \frac{\sin 2\pi/4}{2\pi}, \frac{\sin 3\pi/4}{3\pi} \right\}$$

By L'Hospital Rule

$$n=0, \quad \frac{\cos \pi/4}{4} = 1/4$$

$$\hat{h}[n] = \{0.075, 0.1592, 0.2251, 0.25, 0.2251, 0.1592, 0.075\}$$

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b) The filter designed in part (a) has infinite bandwidth at freq domain and it has gibbs effect.

The freq-response of the designed filter is the convolution of the desired filter with a sinc function.

c) $w[n] = \{ 1/4, 2/4, 3/4, 1, 3/4, 2/4, 1/4 \}$

$h[n]$ is the same function that we found in part (a)

$$\hat{h}_2[n] = h[n] \cdot w[n] = \hat{h}[n] \cdot w[n] = \{0.0188, 0.0786, 0.1688, \underset{\uparrow n=0}{0.25}, 0.1688, 0.0786, 0.0188\}$$

Q3) Given the input $x[n]=\{\ldots, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, 1, 2, 2, 2, 2, \ldots\}$

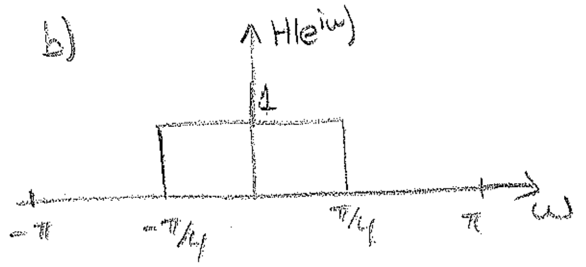
Decimate $x[n]$ by a factor of 4:

a) What should be the cut-off frequency of the low-pass filter that should be used?

b) Design a 3rd order low-pass FIR filter and find the decimated signal. (If you cannot design the filter, use the following filter $h[n]=\left\{\frac{1}{4}, \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}}, \frac{1}{4}\right\}$ to get partial credit)

a) The cutoff freq of the LPF should be $\pi/4$

b)



$$\Rightarrow h[n] = \int_{-\pi/4}^{\pi/4} e^{-jwn}\, dw = \frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n}$$

(Rectangular window) $\Rightarrow w[n] = \{1, \underset{\substack{\uparrow \\ n=0}}{1}, 1\} \Rightarrow h_{lp}[n] = h[n].w[n] = \left\{\frac{\sin -\pi/4}{-\pi}, 1/4, \frac{\sin \pi/4}{\pi}\right\}$

$$= \left\{\frac{1}{\sqrt{2}\pi}, 1/4, \frac{1}{\sqrt{2}\pi}\right\}$$

or

(Triangular window) $w[n] = \{1/2, 1, 1/2\} \Rightarrow h_{lp}[n] = \left\{\frac{1}{2\sqrt{2}\pi}, 1/4, \frac{1}{2\sqrt{2}\pi}\right\}$

I will use rectangular window to filter the signal

$$\hat{x}[n] = x[n] * h_{lp}[n] = \{\ldots 0.7002, 0.7002, \underset{\substack{\uparrow \\ n=3}}{0.9252}, 1.1752, 1.4003, 1.4003 \ldots\}$$

$$= \sum_{n=-\infty}^{\infty} x[k].h[n-k]$$

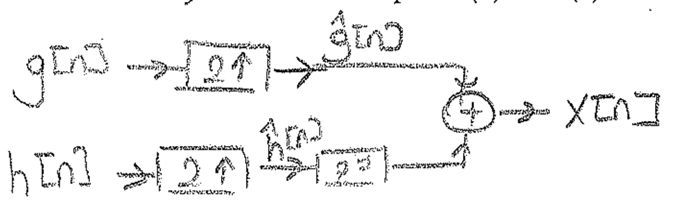
Q4) Let the 4-point DFT of $g[n]=[a,\ c,\ e,\ \alpha]$ be $G[k]=[1,\ 1,\ 1,\ ,1]$ and the 4-point DFT of $h[n]=[b,\ d,\ f,\ 0]$ be $H[k]=[3,\ -i,\ 1,\ i]$.

a) Find the 8-point DFT $X[k]$ of $x[n]=[a,\ b,\ c,\ d,\ e,\ f,\ \alpha,\ 0]$ using $G[k]$ and $H[k]$.

b) What is the advantage of computing 8-point DFT instead of a 7-point DFT? Determine the number of required multiplications to compute 7-point DFT and 8-point FFT.

c) Find $g[n]$ and $h[n]$.

Note that you can answer parts (b) and (c) even if you cannot solve part (a).



$$X[k]=\sum_{n=0}^{7} x[n]\cdot e^{-j2\pi nk/8} = \underbrace{\sum_{n=0,2,4,6} x[n]\cdot e^{-j2\pi nk/8}}_{A} + \underbrace{\sum_{n=1,3,5,7} x[n]\cdot e^{-j2\pi nk/8}}_{B}$$

$$A \Rightarrow \sum_{l=0}^{3} x[2l]\cdot e^{-j2\pi 2lk/8} = \sum_{l=0}^{3} x[2l]\cdot e^{-j2\pi lk/4} = \sum_{l=0}^{3} g[l]\cdot e^{-j2\pi lk/4} = G[k]$$

$\boxed{n=2l}$

$$B \Rightarrow \sum_{l=0}^{3} x[2l+1]\cdot e^{-j2\pi(2l+1)k/8} = \sum_{l=0}^{3} h[l]\cdot e^{-j2\pi lk/4}\cdot\left(e^{-j2\pi k/8}\right) = H[k]\cdot e^{-j2\pi k/8}$$

$n=2l+1$

$\Rightarrow X[k] = G[k] + H[k]\cdot e^{-j2\pi k/8}$, $k=0,1,2\ldots 7$.

$X[4] = G[0] + H[0]\, e^{-j2\pi 4/8}$ ✓ ...

b) We need N^2 complex multiplications to compute 7 point DFT = 49 complex multiplications

We need $\frac{N}{2}(\log N)$ complex multiplications to compute 8 point FFT $= \frac{8.3}{2} = \frac{24}{2}$ multip.

Therefore using point FFT decreases the computational complexity.

c) $g[n] = \frac{1}{4} \sum_{k=0}^{3} e^{j2\pi kn/4}$

$$\Rightarrow g[0] = \frac{1}{4} \cdot 4 = 1$$
$$g[1] = \frac{1}{4} \cdot (1 + i - 1 - i) = 0$$
$$g[2] = \frac{1}{4} \cdot (1 - 1 + 1 - 1) = 0$$
$$g[3] = \frac{1}{4} (1 - i - 1 + i) = 0$$

$g[n] = [1, 0, 0, 0]$

$h[n] = \frac{1}{4} \sum_{k=0}^{3} H[k] \cdot e^{j2\pi kn/4}$

$3 \quad -i \quad 1 \quad i$

$$\Rightarrow h[0] = \frac{1}{4} \cdot (3 - i + 1 + i) = 1$$
$$h[1] = \frac{1}{4} (3 + 1 - 1 + 1) = 1$$
$$h[2] = \frac{1}{4} (3 + i + 1 - i) = 1$$
$$h[3] = \frac{1}{4} (3 - 1 - 1 - 1) = 0$$

$h[n] = [1, 1, 1, 0]$

Important Formulas

CTFT

$$X_a(j\Omega)=\int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt$$

ICTFT

$$x_a(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t}d\Omega$$

DTFT

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

IDTFT

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DFT

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \qquad 0 \le k \le N-1$$

IDFT

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{\frac{j2\pi}{N}kn} \qquad 0 \le n \le N-1$$