Q1) Let $x_c(t) = \sin\left(2\pi 500 t + \frac{\pi}{2}\right) + \cos(2\pi 1000 t)$

This signal is sampled with $f_s = 8\,KHz$: $x[n] = x_c\left(\frac{n}{f_s}\right)$

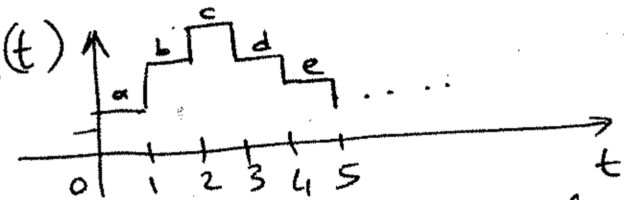
a) Plot $|X(e^{jw})|$ $\left(X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} \right)$

b) We have $n = 0, 1, \ldots, 1023$ samples of $x[n]$.

Approximately plot $|X[k]|$ $k = 0, 1, \ldots, 1023$.

($X[k]$ is the 1024 point DFT of $x[n]$).

c) Let $x_d(t)$



Is $x_d(t)$ a band-limited signal? Explain your answer!

d) Let $x[n] = x_d(n \cdot \frac{1}{2}) = \{\ldots, a, a, b, b, c, c, d, d, \ldots\}$

Can we reconstruct $x_d(t)$ from $x[n]$? Explain your answer!

a) $f_s = 8\,kHz$ is mapped to $w = 2\pi$

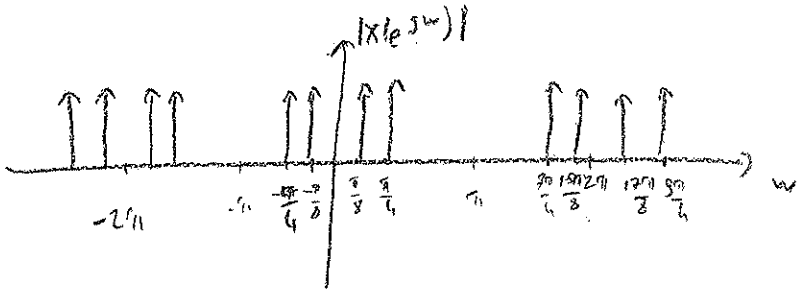
thus, $500\,Hz$ is mapped to $w = \frac{\pi}{8}$

$-500\,Hz$ is mapped to $w = -\frac{\pi}{8}$

$1\,kHz$ is mapped to $w = \frac{\pi}{4}$

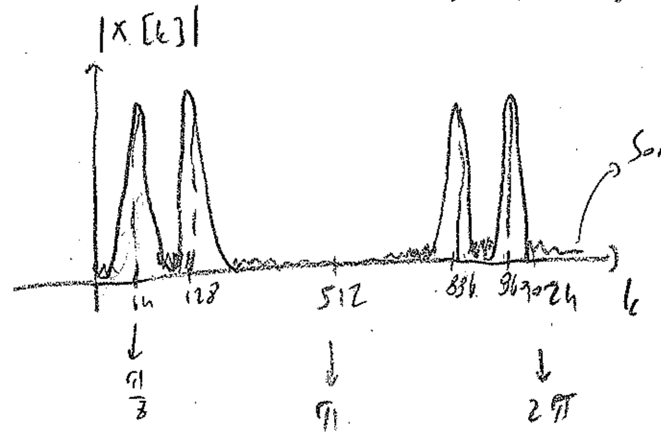
$-1\,kHz$ is mapped to $w = \frac{-\pi}{4}$

impulsive components.



b) $X[k]$ will have peaks at indices corresponding to 500 Hz, -500 Hz, 1 kHz, -1 kHz

But peaks will not be impulsive, they will look like sinc's



$\frac{512}{4} = 128$

1024 corresponds to 2π $\Rightarrow$ 64 corresponds to $\frac{\pi}{8}$

1024 " " " $\Rightarrow$ 128 " to $\frac{\pi}{4}$

1024 corresponds to 2π $\Rightarrow$ 960 corresponds to $-\frac{\pi}{8}$

1024 corresponds to 2π $\Rightarrow$ 896 corresponds to $-\frac{\pi}{4}$

c) This is not a band limited signal because it has discontinuities (Jumps). Such a Jump requires infinite frequency.

d) $x[n] = x_d[\frac{n}{2}]$

Although the signal is not bandlimited, we CAN reconstruct it from $x[n]$, because we know that $x[n]$ is a staircase function. If it had an arbitrary form, the answer would be No. However, that is not the case.

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Q2

a) Perform the 4-point circular convolution of

$$x[n] = \{\underset{\uparrow\, n=0}{1}, 2, 1\} \quad \text{and} \quad h[n] = \{\underset{\uparrow}{1}, 2, 2\}$$

$$w[n] = x[n] \; ④ \; h[n]$$

b) Calculate $w[n]$ using D.F.T.

$$w[n] = x[n] \; ④ \; h[n] = \sum_{l=0}^{2} h[l]\, x[(n-l)_4]$$

$$\Rightarrow w[0] = \sum_{l=0}^{2} h[l]\, x[(-l)_4] = h[0]\,x[0] + \overbrace{h[1]\,x[3]}^{0} + h[2]\,x[2] = 3$$

$$w[1] = \sum_{l=0}^{2} h[l]\, x[(1-l)_4] = h[0]\,x[1] + h[1]\,x[0] + h[2]\,\underset{0}{x[3]} = 4$$

$$w[2] = \sum_{l=0}^{2} h[l]\, x[(2-l)_4] = h[0]\,x[2] + h[1]\,x[1] + h[2]\,x[0] = 1+4+2 = 7$$

$$w[3] = \sum_{l=0}^{2} h[l]\, x[(3-l)_4] = h[0]\,\underset{0}{x[3]} + h[1]\,x[2] + h[2]\,x[1] = 4+2 = 6$$

$$\Rightarrow w[n] = \{\underset{\uparrow\, n=0}{3}, 4, 7, 6\}$$

b) $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$, $N=4$

$\Rightarrow X[0] = 4$

$X[1] = x[0] + x[1] e^{-j\frac{2\pi}{4}} + x[2] e^{-j\frac{2\pi \cdot 2}{4}} = 1 + 2 e^{-j\frac{\pi}{2}} + \cancel{1 \cdot e^{-j\pi}}$
$= -2j$

$X[2] = x[0] + x[1] e^{-j\frac{2\pi \cdot 2}{4}} + x[2] e^{-j\frac{2\pi 4}{4}} = 2 + 2 \cdot e^{-j\pi} = 0.$

$X[3] = x[0] + x[1] e^{-j\frac{2\pi 3}{4}} + x[2] e^{-j\frac{2\pi 3 \cdot 2}{4}} = 1 + \underbrace{2 \cdot e^{-j\frac{3\pi}{2}}}_{+2j} + \underbrace{e^{-j3\pi}}_{-1} = 2j$

$\Rightarrow X[k] = \{4, -2j, 0, 2j\}$

$H[k] = \sum_{n=0}^{N-1} h[n] e^{-j\frac{2\pi kn}{N}}$, $N=4$

$\Rightarrow H[0] = 5$

$H[1] = h[0] + h[1] e^{-j\frac{2\pi}{4}} + h[2] e^{-j\frac{4\pi}{4}} = 1 - 2j + 2(-1) = -1 - 2j$

$H(2) = h[0] + h[1] \underbrace{e^{-j\frac{4\pi}{4}}}_{-1} + h[2] \underbrace{e^{-j\frac{8\pi}{4}}}_{1} = 1.$

$H(3) = h[0] + h[1] e^{-j\frac{2\pi 3}{4}} + h[2] e^{-j\frac{4\pi \cdot 3}{4}} = 1 + 2j - 2 = -1 + 2j$

$H[k] = \{5, -1-2j, 1, -1+2j\}$

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$\Rightarrow W[k] = H[k] X[k] = \{20, -4+2j, 0, -4-2j\}$

$w[n] = \frac{1}{4} \sum_{k=0}^{3} W[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{4} \sum_{k=0}^{3} W[k] e^{j\frac{\pi}{2}kn}.$

$\Rightarrow w[0] = \frac{1}{4}(12) = 3.$

$w[1] = \frac{1}{4}\left(20 + (-4+2j) \cdot j + 0 + (-4-2j) \cdot \underbrace{e^{j\frac{3\pi}{2}}}_{-j}\right) = \frac{1}{4}(20 - 4j - 2 + 4j - 2)$
$= 4$

$w[2] = \frac{1}{4}\left(20 + e^{j\pi}(-4+2j) + 0 + (-4-2j) e^{j3\pi}\right) = \frac{1}{4}(20 + 4 - 2j + 4 + 2j) = 7$

$w[3] = \frac{1}{4}\left(20 + \underbrace{e^{j\frac{3\pi}{2}}}_{-j} (-4+2j) + 0 + (-4-2j) \underbrace{e^{j\frac{9\pi}{2}}}_{j}\right) = \frac{1}{4}(20 + 4j + 2 - 4j + 2)$
$= 6.$

$\Rightarrow w[n] = \{\overset{n=0}{\underset{\uparrow}{3}}, 4, 7, 6\}$

Q3 Let $x_c(t)$ be a bandlimited signal with BW = 4KHz. This signal is sampled with $f_s = 8KHz$. We want to high-pass filter $x[n] = x_c(nT_s)$, $n = 0, \mp 1, ..$ with cut-off $\Omega_c = 3$ KHz (Actual frequency).

a) What should be the corresponding normalized frequency w_c corresponding to Ω_c ?

b) Design a recursive IIR high-pass filter using a Butterworth analog prototype with 3dB cut-off at w_c.

Butterworth filter: $|H_a(j\Omega)|^2 = \dfrac{1}{1+(j\Omega/j\Omega_c)^2}$

Use bilinear transformation.

a) $f_s = 8kHz \Rightarrow w = 2\pi$ corresponds to $8kHz \Rightarrow W_c = \frac{3\pi}{4}$ corresponds to $3kHz$

b) $N=1$, $H_a(s) = \dfrac{\Omega_s}{s+\Omega_s}$ → cut off = Ω_s this is the low pass Butterworth filter, we can design a highpass using this one.

$\Omega = \frac{2}{T} \tan \frac{w}{2}$ by bilinear transform

Thus, $\Omega = \frac{2}{T} \cdot \tan \frac{3\pi}{8} \cong \frac{4.828}{T}$

$\Rightarrow$ Our analog low pass prototype is $\dfrac{\frac{4.828}{T}}{s + \frac{4.828}{T}}$

Let's digitalize it; $s = \frac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$

$$\Rightarrow H_{e_1}(z^{-1}) = \frac{\frac{4.828}{T}}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\frac{4.828}{T}} = \frac{2.414}{\frac{1-z^{-1}}{1+z^{-1}}+2.414} = 2.414 \frac{1+z^{-1}}{3.414+1.414z^{-1}}$$

$$H(z) = \frac{Y(z)}{X(z)} = 2.414 \frac{1+z^{-1}}{3.414+1.414z^{-1}} \Rightarrow 3.414\, Y(z) + 1.414\, Y(z)z^{-1} = 2.414\, X(z) + 2.414\, X(z)z^{-1}$$

take inverse z transform

$\to 3.414\, y[n] + 1.414\, y[n-1] = 2.414\, x[n] + 2.414\, x[n-1]$.

Let's find the impulse response, $x[n] = \delta[n]$

$n=0 \Rightarrow h[0] = \frac{2.414}{3.414} = 0.707$

$n=1 \Rightarrow 3.414\, h[1] + 1.414\, h[0] = 2.414 \Rightarrow h[1] = \left(2.414 - \frac{2.414}{3.414}\cdot 1.414\right)/3.414$

$= 0.414$

$n=2 \Rightarrow 3.414\, h[2] = -1.414\, h[1]$,

$\Rightarrow h[2] = -0.414\, h[1] = -(0.414)^2$

$h[n] = -0.414\, h[n-1] = (0.414)^n(-1)^{n+1}$

Thus, the impulse response $h[n]$ is, $h[n] = \begin{cases} 0.707, & n=0 \\ (0.414)^n(-1)^{n+1}, & n>0. \end{cases}$

This is the low pass Butterworth. To obtain high pass,

$$h_{hp}[n] = \delta[n] - h[n] = \begin{cases} 0.293, & n=0 \\ (-1)^n(0.414)^n, & n>0 \end{cases}$$

does the job.

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Q4// Let $y[n] = 0.9\, y[n-1] + x[n]$

where $x[n]$ is white, zero-mean r.p. with $\sigma_x^2 = 1$.

a) Calculate $r_y[n]$.

b) Determine the spectrum $S_y(e^{j\omega})$.

c) Determine the first order L.M.M.S.E. predictor for the random process $y[n]$.

d) Let $x[n] = \{1, -1, 1, -1, -1, 1, -1, -1, -1, 1, -1, 1\}$

Can $x[n]$ be a realization of a white, zero-mean random process? Justify your answer

a) $r_y[0] = E[y[n]y[n]] = E[(0.9y[n-1] + x[n])(0.9y[n-1] + x[n])]$

$= 0.81\, \underbrace{E[y[n-1]y[n-1]]}_{r_y[0]} + 1.8\, E[y[n-1]x[n]] + \underbrace{E[x[n]x[n]]}_{\sigma_x^2}$

$E[y[n-1]x[n]] \underset{i}{=} E[y[n-1]]\,E[x[n]] = 0$

by independence. $y[n-1]$ is a casual system, it is not dependent on a future value ($x[n]$)

$\Rightarrow r_y[0] = 0.9^2 r_y[0] + 1 \Rightarrow 0.19\, r_y[0] = 1 \Rightarrow \boxed{r_y[0] \cong 5.26}$

$r_y[1] = E[y[n]y[n-1]] = E[0.9\, y[n-1]y[n-1] + x[n]y[n-1]]$

$= 0.9\, E[y[n-1]y[n-1]] + \underbrace{E[x[n]y[n-1]]}_{0} = \boxed{0.9\, r_y[0]}$

$r_y[2] = E[y[n]y[n-2]] = E[[0.9y[n-1] + x[n]]y[n-2]] = 0.9\, E[y[n-1]y[n-2]] + E[y[n-2]x[n]]$

$= 0.9\, r_y[1]$

$\Rightarrow r_y[n] = 5.26 \cdot (0.9)^n$ for $n \geq 0$

$r_y[n] = r_y[-n]$ by wss

b) $S_y(e^{jw}) = \sum_{n=-\infty}^{\infty} r_y[n] e^{-jwn} \overset{\uparrow}{=} 2 \sum_{n=0}^{\infty} r_y[n] e^{-jwn} - r_y[0] e^{-jw0}$

$= 2 \sum_{n=0}^{\infty} 5.26 . (0.9)^n e^{-jwn} - 5.26$

$= 10.52 \sum_{n=0}^{\infty} (0.9e^{-jw})^n - 5.26 = 10.52 \frac{1}{1-0.9e^{-jw}} - 5.26$

$= 5.26 \left(\frac{2}{1-0.9e^{-jw}} - 1 \right) = 5.26 . \frac{1+0.9e^{-jw}}{1-0.9e^{-jw}}$

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c) 1st order LMMSE $\Rightarrow \hat{y}[n] = a\, y[n-1]$.

min $E[(y[n]-\hat{y}[n])^2] \Rightarrow E[\frac{d}{da}(y[n]-\hat{y}[n])^2] = 0 \Rightarrow E[2(y[n]-\hat{y}[n])(-1)y[n-1]] = 0$

$\Rightarrow E[(y[n]-\hat{y}[n])\, y[n-1]] = 0 \Rightarrow E[y[n]\, y[n-1]] = E[\hat{y}[n]\, y[n-1]]$

$\Rightarrow r_y[1] = E[a\, y[n-1] y[n-1]] \Rightarrow r_y[1] = a\, r_y[0] \Rightarrow a = \frac{r_y[1]}{r_y[0]}$.

Thus, $a = 0.9$, our predictor is $\hat{y}[n] = 0.9\, y[n-1]$

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d) The sample mean of $x[n]$ is $\tilde{m}_x = \frac{1}{12} \sum_{i=0}^{11} x[i] = 0$.

The sample variance is $\tilde{\sigma}_x = \frac{1}{12} \sum_{i=0}^{11} (x[i]-0)^2 = 1. = \tilde{r}_y[0]$

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$\tilde{r}_y[1] = \sum_{i=1}^{11-1} r_y[i]\, r_y[i-1] = -1 -1 -1 +1 -1 -1 -1 -1 -1 -1 -1 = -9$

$\tilde{r}_y[2] = -1 +1 -1 -1 +1 +1 +1 +1 +1 +1 = 4$

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$\vdots$

This can be a realization of a white zero mean random process.
The sample mean and sample variance comply with the zero mean.
For the process to be white, we need $r_y[n] = \sigma^2 \delta[n]$.
This is not actually the case for this sequence, but it is only a realization. The sample correlation may not be equal to the real correlation, so it doesn't falsify being a realization of white 0-mean process

Q5)

a) What is the pitch period?

b) Describe the L.M.M.S.E based speech modeling. Which systems do use the L.M.M.S.E based speech coding?

c) What is the bit-rate of PCM quantizer when the speech is sampled at 8 KHz in land-line telephone systems?

d) Let $x[n] = \{1, 0, -1, 1, 0, -1, 1, 0, -1, 1, 0, -1\}$

Calculate the period of $x[n]$ using the AMDF function (or autocorrelation fuction). Which one is better? Why?

a) In speech, the voiced sounds are (nearly) periodic signals. The pitch period is the period of those voiced sounds.

b) In LMMSE based speech modeling, filter coefficients that minimize the mean square error are transmitted, rather than actual speech samples. LPC-10 uses LMMSE based speech modelling. Additionally, GSM also uses an advanced version of this. pitch period?

c) PCM quantizer operates at 64 kBps. 8000 samples are processed in a second, 8 bit quantizer is used for each sample.

d) $AMDF(k) = \frac{1}{N} \sum_{i=0}^{N-1-k} |x[i] - x[i+k]|$

$$AMDF(1) = \frac{1}{12}\Big\{ |x[1]-x[0]| + |x[2]-x[1]| + \cdots + |x[11]-x[10]| \Big\}$$

$$= \frac{1}{12} \cdot 14 = \frac{14}{12}$$

$$AMDF(2) = \frac{1}{12} \sum_{i=0}^{9} |x[i+2] - x[i]| = \frac{1}{12} \{|x[2]-x[0]| + |x[3]-x[1]| -- + |x[11]-x[9]|\}$$

$$= \frac{1}{12} \cdot [\ 2+1+2+2+1+1+2+1+1+2\ ] = \frac{15}{12}$$

$$AMDF(3) = \frac{1}{12} \sum_{i=0}^{8} |x[i+3] - x[i]| = \frac{1}{12}[\ |x[3]-x[0]| + \text{———} + |x[11]-x[8]|\ ]$$

$$= \frac{1}{12}[\ 0 + 0 + \text{———} + 0\ ] = 0.$$

$k=3$ minimizes the AMDF, thus period is 3.

If we want to consider the autocorrelation, we will try to maximize it, and $k=3$ will again maximize it.

$\frac{1}{N} \sum_{i=1}^{N-k-3} rx[i]\, rx[i+3]$ will be the maximum.

AMDF is better, because it is much more efficient. For calculating AMDF we only have additions, but for autocorrelation, we will need many multiplications.

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