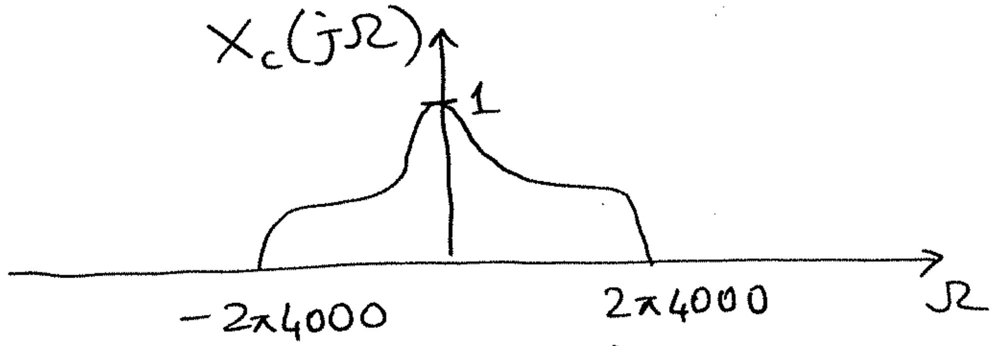
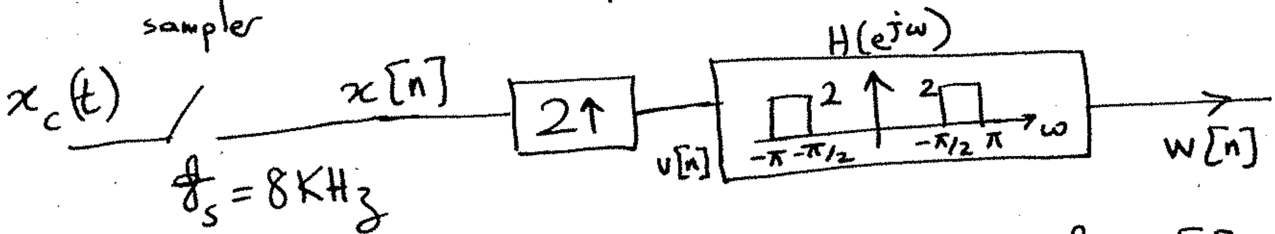
Q1

Let $x_c(t) \overset{\mathcal{F}}{\longleftrightarrow} X_c(j\Omega)$ be a C.T.F.T pair with





a) Plot $X(e^{j\omega})$ which is the D.T.F.T. of $x[n]$.

b) Plot $V(e^{j\omega})$ " " " " " $v[n]$.

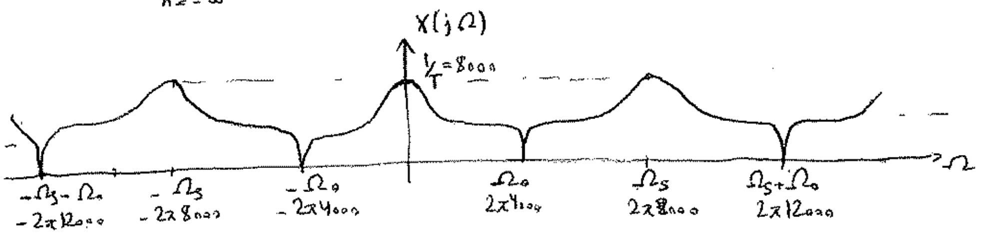
c) Plot $W(e^{j\omega})$ " " " " " $w[n]$.

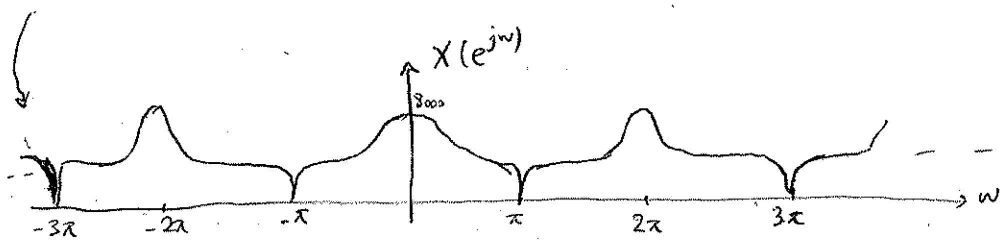
Explain your work and plots.

$\Omega_s = 2\pi 8000$
$\Omega_0 = 2\pi 4000$

a) $x_p(t) = x(n) = x(nT_s)$ $\qquad X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \Rightarrow x[n] = x(t) * p(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s) \Rightarrow X(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\left(j(\Omega - k\Omega_s)\right)$

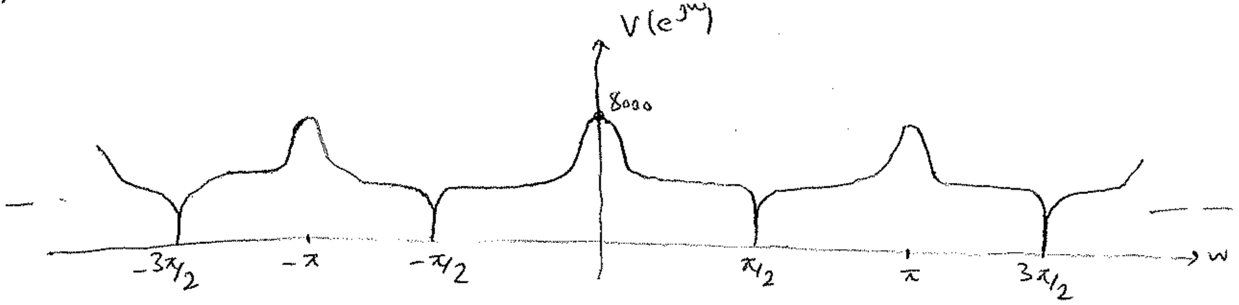




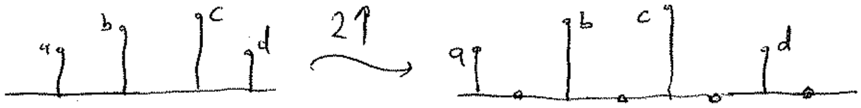
b) upsample : $x[n] \to \boxed{2\uparrow} \to v[n]$

$$v[n] = \begin{cases} x[n/2] & n = \text{even} \\ 0 & \text{o.w.} \end{cases}$$

$$V(e^{j\omega}) = X(e^{j2\omega})$$

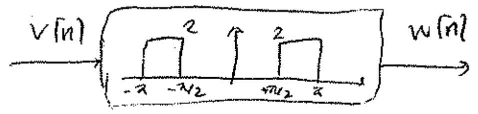


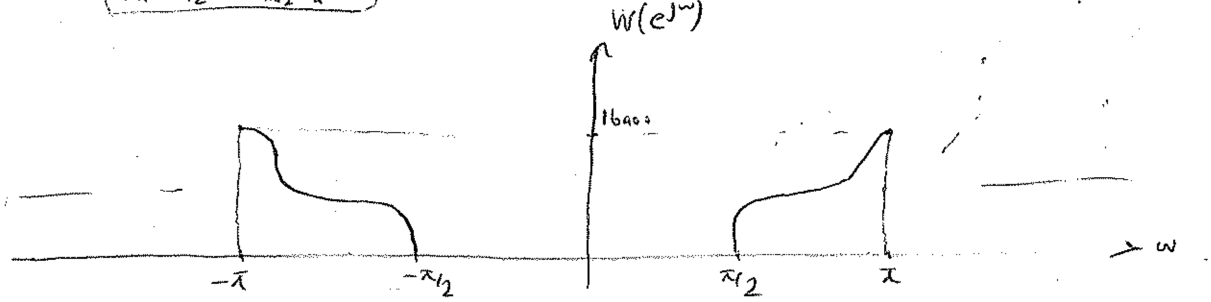
in upsampling we add $(M-1)$ zeros between 2 samples :



$$V(e^{j\omega}) = \sum_{n=-\infty} x[n/2]\, e^{-j\omega n} = \sum_{l=-\infty}^{\infty} x[l]\, e^{-j(2\omega)l} = X(e^{j2\omega})$$

c)





using normal convolution: $\{1, \underset{\uparrow n=0}{3}, 3, 1\} = y[n]$

Q2 Let $x[n] = \{1, \underset{\substack{\uparrow \\ n=0}}{2}, 1\}$, $v[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 1\}$

Calculate $y[n] = x[n] * v[n]$ using D.F.T.

Show your work to get partial credit!

$y[n] = x[n] * v[n] \xrightarrow{N\text{-DFT}} Y[k] = X[k]V[k]$

$L = \text{length } x = 3$
$M = \text{length } v = 2$ $\Rightarrow N \geq M + L - 1$

$\Rightarrow N \geq 4 \Rightarrow \boxed{N=4}$

to avoid aliasing we should use $(M+L-1)$-point DFT, which here is $N=4$:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4}kn} = 2 + 1e^{-j\frac{\pi k}{2}} + 0 + 1e^{-j\frac{3\pi k}{2}}$$

periodic $\leftarrow x[n] = \{\underset{\uparrow n=0}{2}, 1, 0, 1\}$

$$= 2 + e^{-j\frac{\pi k}{2}} + e^{j\frac{\pi k}{2}} = \boxed{2 + 2\cos\left(\frac{\pi k}{2}\right) = X[k]}$$

$\Rightarrow \boxed{X[k] = 2 + 2\cos\left(\frac{\pi k}{2}\right) = \{4, 2, 0, 2\}}$

$$V[k] = \sum_{n=0}^{3} v[n] e^{-j\frac{\pi}{2}kn} = 1 + e^{-j\frac{\pi k}{2}}$$

$v[n] = \{1, 1, 0, 0\}$

$\Rightarrow V[k] = 1 + e^{-j\frac{\pi k}{2}} = 1 + \cos\left(\frac{\pi k}{2}\right) - j\sin\left(\frac{\pi k}{2}\right) = \boxed{\{2, 1-j, 0, 1+j\}} = V[k]$

$$Y[k] = X[k] \cdot V[k] = \{8, 2-j2, 0, 2+j2\}$$

$$y[n] = \frac{1}{N}\sum_{n=0}^{N-1} Y[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{4}\sum_{k=0}^{3} Y[k] e^{j\frac{\pi}{2}kn} = \frac{1}{4}\left[8 + (2-j2)e^{j\frac{\pi}{2}n} + 0 + (2+j2)e^{j\frac{3\pi}{2}n}\right]$$

$\boxed{y[n] = \{\underset{\uparrow n=0}{3}, 3, 1, 1\} \xrightarrow{\text{periodic}} y[n] = \{1, \underset{\uparrow n=0}{3}, 3, 1\}}$

$n=0 \Rightarrow y[0] = \frac{1}{4}(8 + 2 - \cancel{j2} + 0 + 2 + \cancel{j2}) = 3$ / $n=1 \Rightarrow y[1] = \frac{1}{4}(8 + 2 + j2 + 0 + 2 - j2) = 3$

$n=2 \Rightarrow y[2] = \frac{1}{4}(8 + (-2 + j2) + 0 + (-2 - j2)) = 1$ / $n=3 \Rightarrow y[3] = \frac{1}{4}(8 + (-2 - \cancel{j}) + 0 + 2 + j\cancel{2}) = 1$

$X[k] = 2 + e^{-j\frac{2\pi k}{4}} = \{3, 2-j, 1, 2+j\}$ ⟶ using direct DFT relation

Q3

Given $x[n] = \{\underset{\substack{\uparrow \\ n=0}}{2}, 1, 0, 0\}$

a) Calculate the 4-point DFT of $x[n]$ using the Decimation-in-time FFT algorithm.

b) Calculate the 4-point DFT of $x[n]$ using the Decimation-in-frequency FFT algorithm.

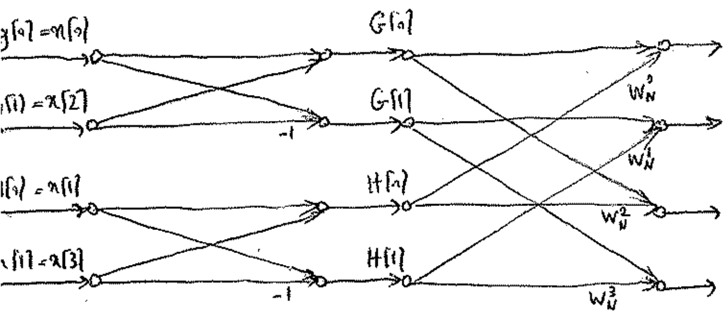
Draw the block-diagrams of D.I.T (Deci. in Time) and D.I.F. (Deci. in Freq.) algorithms and clearly show your work. Otherwise you will get zero credit!

a) $X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} = \sum_{\substack{n=0 \\ \text{even}}}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} + \sum_{\substack{n=0 \\ \text{odd}}}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$

$= \sum_{l=0}^{\frac{N}{2}-1} x[2l]\, \underbrace{W_N^{2lk}} + \sum_{l=0}^{\frac{N}{2}-1} x[2l+1]\, \underbrace{W_N^{(2l+1)k}} = \sum_{l=0}^{N/2-1} x[2l]\, W_{N/2}^{kl} + W_N^k \sum_{l=0}^{N/2-1} x[2l+1]\, W_{N/2}^{kl}$

$W_N^{2lk} = e^{-j\frac{2\pi}{N}(2lk)} = e^{-j\frac{2\pi}{(N/2)}kl} = W_{N/2}^{kl}$ $\qquad W_N^{(2l+1)k} = W_N^k\, W_{N/2}^{kl}$

$X[k] = G[k] + W_N^k\, H[k]$ $\qquad g[n] = x[2l]$ $\qquad h[n] = x[2l+1]$

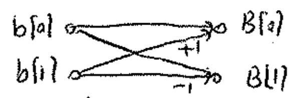


$X[0] = G[0] + W_4^0\, H[0]$

$X[1] = G[1] + W_4^1\, H[1]$

$X[2] = G[0] + W_4^2\, H[0]$

$X[3] = G[1] + W_4^3\, H[1]$



$B[k] = \sum_{n=0}^{1} b[n]\, e^{-j\pi kn}$

$\Rightarrow \begin{cases} B[0] = b[0] + b[1] \\ B[1] = b[0] - b[1] \end{cases}$

for part (a) and (b)

$G[0] = g[0] + g[1] = x[0] + x[2] = 2$ / $G[1] = g[0] - g[1] = 2$

$H[0] = h[0] + h[1] = x[1] + x[3] = 1$ / $H[1] = [0] - h[1] = 1$

$X[0] = G[0] + H[0] = 3$ / $X[1] = G[1] + e^{-j\frac{2\pi}{4}} H[1] = 2 - j$

$X[2] = G[0] + W_4^2 H[0] = 2 + e^{-j\pi}(1) = 1$ / $X[3] = G[1] + W_4^3 H[1] = 2 + e^{-j\frac{3\pi}{2}}(1) = 2 + j$

$$\Longrightarrow \boxed{X[k] = \{3, 2-j, 1, 2+j\}}$$

Decimation-in-Time!

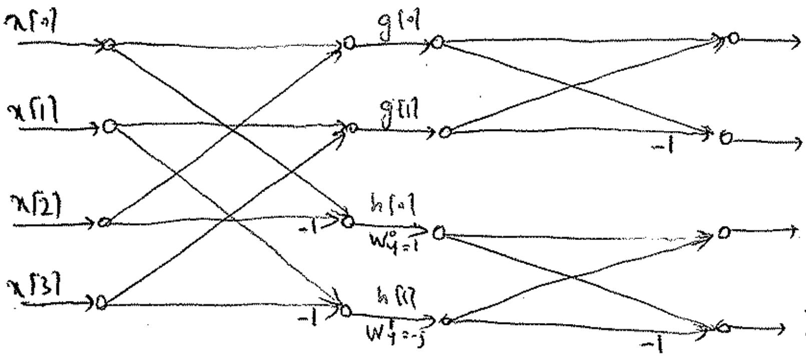
b) $X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn} + \sum_{N/2}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n]\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\, \underbrace{e^{-j\frac{2\pi}{N}(n+\frac{N}{2})k}}_{e^{-j\frac{2\pi}{N}kn}\, e^{-j\frac{2\pi}{N}\frac{N}{2}k} = e^{-j\frac{2\pi}{N}kn}(-1)^k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n]\, e^{-j\frac{2\pi}{N}kn} + (-1)^k \sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}]\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + (-1)^k x[n+\tfrac{N}{2}]\right) e^{-j\frac{2\pi}{N}kn}$$

$$\Longrightarrow \left| \begin{array}{l} X[2k] = \sum_{n=0}^{\frac{N}{2}-1} (x[n] + x[n+\frac{N}{2}])\, W_{N/2}^{kn} = G[k] \Longrightarrow g[n] = x[n] + x[n+\frac{N}{2}] \\ X[2k+1] = \sum_{n=0}^{N/2-1} (x[n] - W_N^n\, x[n+\frac{N}{2}])\, W_{N/2}^{kn} = H[k] \Longrightarrow h[n] = (x[n] - x[n+\frac{N}{2}])\, W_N^n \end{array} \right.$$

$x[n] = \{2, 1, 0, 0\}$



$X[0] = g[0] + g[1]$

$X[2] = g[0] - g[1]$

$X[1] = h[0] + h[1]$

$X[3] = h[0] - h[1]$

$g[0] = 2 = x[0] + x[0+\frac{4}{2}]$ / $g[1] = x[1] + x[3] = 1$

$h[0] = (\overset{2}{x[0]} - \overset{0}{x[2]})\, W_4^0 = 2$ / $h[1] = (x[1] - x[3])\, W_4^1 = 1e^{-j\frac{\pi}{2}} = -j$

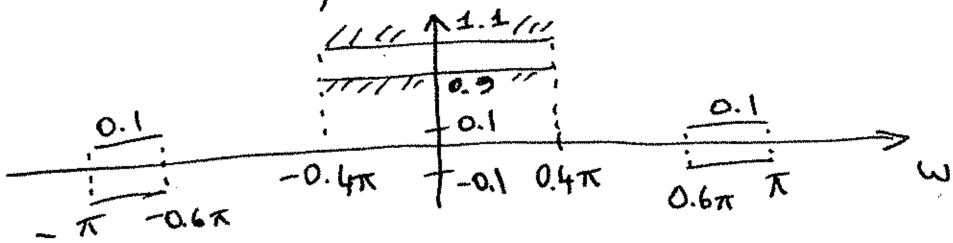
$X[0] = G[0] = g[0] + g[1] = 3$ / $X[1] = h[0] + h[1] = 2 - j = H[0]$

$X[2] = G[1] = g[0] - g[1] = 1$ / $X[3] = h[0] - h[1] = 2 + j = H[1]$

$\Longrightarrow X[k] = \{3,\ 2-j,\ 1,\ 2+j\}$

Q4 Equiripple FIR filter Design:

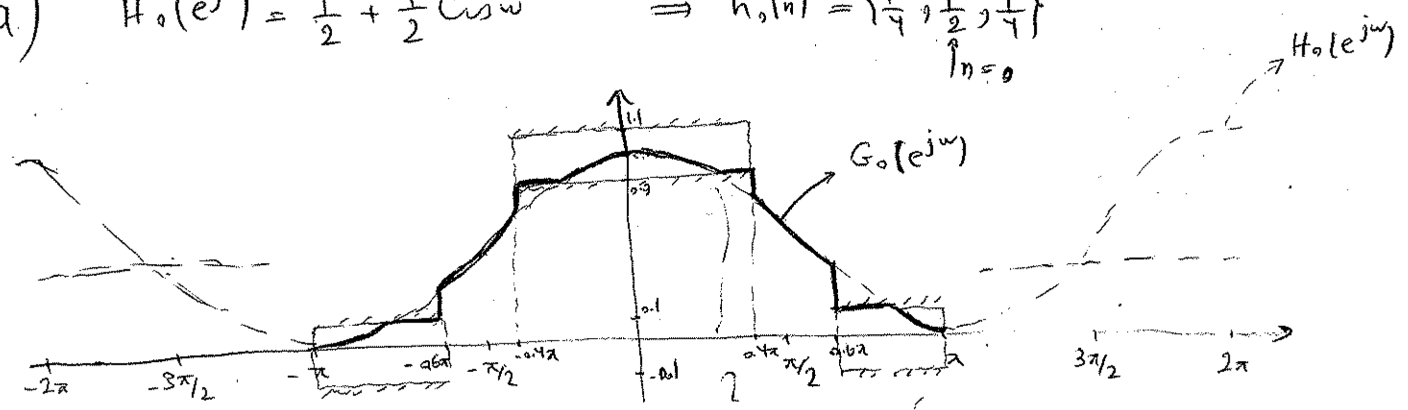
a) Let $H_0(e^{j\omega}) = \frac{1}{2} + \frac{1}{2}\cos\omega$ and the required freq. domain specs are as follows:



Obtain $G_0(e^{j\omega})$ by imposing the above freq. domain specs. on $H_0(e^{j\omega})$ and plot

b) Let $g_6[n] = \{\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\}$ ($\uparrow n=0$) and you want to design a 5-th order zero-phase filter. Determine $h_7[n]$.

a) $H_0(e^{j\omega}) = \frac{1}{2} + \frac{1}{2}\cos\omega \Rightarrow h_0[n] = \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$ ($\uparrow n=0$)



The bounds for frequency domain is as follows:

$$G_0(e^{j\omega}) = \begin{cases} H_0(e^{j\omega}) + E_0(e^{j\omega}) & H_0(e^{j\omega}) + E_0(e^{j\omega}) > H_0(e^{j\omega}) \\ H_0(e^{j\omega}) - E_0(e^{j\omega}) & H_0(e^{j\omega}) - E_0(e^{j\omega}) < H_0(e^{j\omega}) \\ H_0(e^{j\omega}) & \text{O.W.} \end{cases} \longrightarrow \text{freq. domain condition}$$

$$H_0(e^{j\omega}) - E(\omega) \leqslant H_{id}(e^{j\omega}) \leqslant H_0(e^{j\omega}) + E(\omega)$$

b) $h_{K+1}[n] = \begin{cases} g_K[n] & n \in I \\ 0 & \text{o.w.} \end{cases}$; $I = \{-L, \dots, -1, 0, 1, \dots, L\}$

Time-domain condition.

$\boxed{N = 2L+1 = 5 \Rightarrow L = 2}$

$I = \{-2, -1, 0, 1, 2\}$

$h_7[n] = \begin{cases} g_6[n] & n \in \{-2, -1, 0, 1, 2\} \\ 0 & \text{o.w.} \end{cases}$

$\Longrightarrow \quad h_7[n] = \{\frac{1}{8}, \frac{1}{4}, \underset{\uparrow n=0}{\frac{1}{2}}, \frac{1}{4}, \frac{1}{8}\}$

5th order zero-phase filter

$\hookrightarrow h[n] = h[-n]$