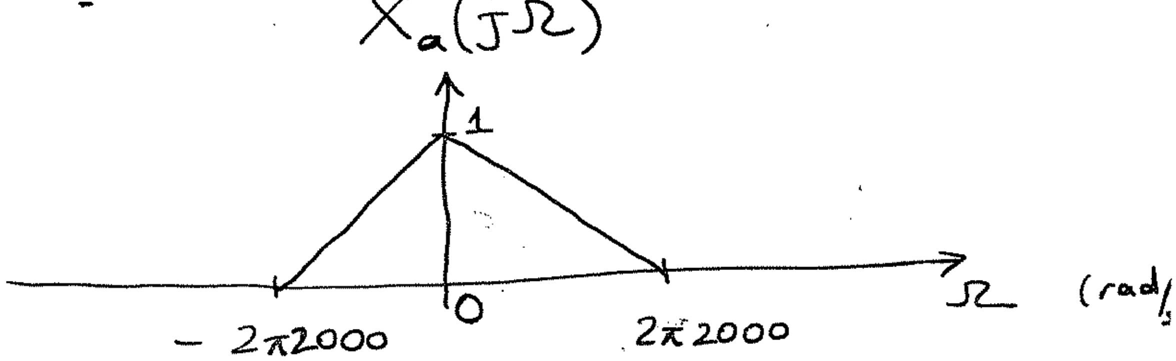
Given $x_a(t)$ with continuous Fourier Transform:



a) $x_a(t) \longrightarrow \otimes \longrightarrow x_p(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t-nT_s)$

$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$ $\qquad f_s = \frac{1}{T_s} = 8$ KHz.

Plot $X_p(j\Omega)$.

b) Let the sampling frequency be $f_{sd} = 4$KHz.

Plot $X_{pd}(j\Omega)$ $\quad$ ($x_a(t) \longrightarrow \otimes \longrightarrow x_{pd}(t)$, with $\sum_n \delta(t-nT_{sd})$, $T_{sd} = \frac{1}{f_{sd}}$)

c) Plot the D.T.F.T. of $x[n] = \{x(nT_s)\}_{n=-\infty}^{\infty}$

d) Plot the D.T.F.T. of $x_d[n] = \{x(nT_{sd})\}_{n=-\infty}^{\infty}$.

e) Can you obtain $x_d[n]$ from $x[n]$? If yes, draw the block-diagram of your system obtaining $x_d[n]$ from $x[n]$. If no, explain!

f) Can you obtain $x[n]$ from $x_d[n]$? If yes, draw the block-diagram of your system obtaining $x[n]$ from $x_d[n]$. If no, explain your answer!

2 Let $x_c(t) = \cos 2\pi 450 t$. This signal is sampled with $f_s = 1.5$ KHz:

$$x[n] = x_c(nT_s), \quad n = 0, \mp 1, \dots \quad ; \quad T_s = \frac{1}{f_s}.$$

We have 512 samples of $x[n]$.

a) Plot $X(e^{j\omega})$ of $x[n]$, $n = 0, \mp 1, \mp 2, \dots$

b) Approximately plot the DFT magnitude: $|X[k]|$ (the DFT size is $N = 512$).

Solutions

Q1.

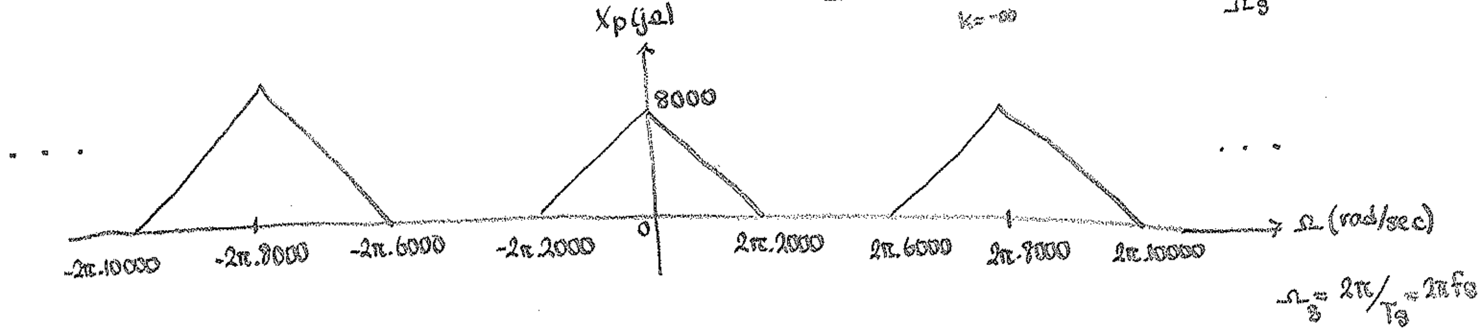
a) $x_p(t) = x_a(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow{\text{CTFT}} X_p(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(j\left(\Omega - \frac{2\pi k}{T_s}\right)\right)$

$f_s = \frac{1}{T_s} = 8\text{kHz}$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - \frac{2\pi k}{T_s}\right)\right)$$

put T_s in place

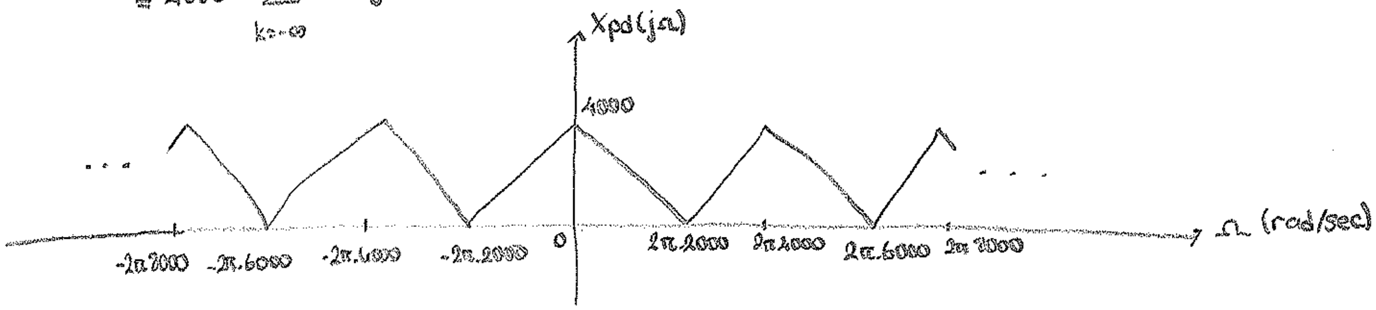
$$\overset{\downarrow}{=} 8000 \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k.\underbrace{2\pi.8000}_{\Omega_s}))$$



$\Omega_s = 2\pi/T_s = 2\pi f_s$

b) $X_{pd}(j\Omega) = \frac{1}{T_{sd}} \sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - k.\frac{2\pi}{T_{sd}}\right)\right)$

$$= 4000 \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k.2\pi.4000))$$

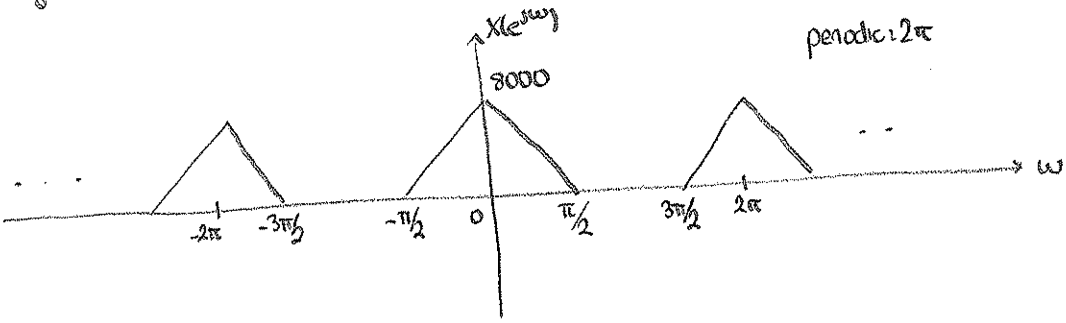


c) $X_p(j\Omega) = \int_{-\infty}^{\infty} x_p(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) e^{-j\Omega t} dt = \sum_{n=-\infty}^{\infty} x_a(nT_s) e^{-j\Omega T_s . n}$ (1)

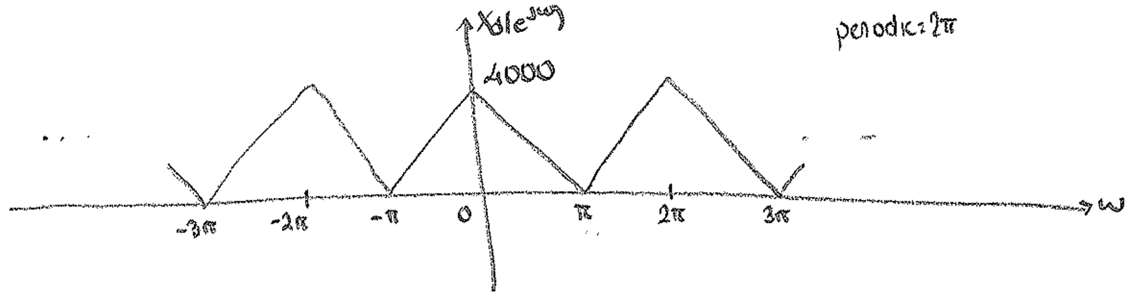
$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_a(nT_s) e^{-j\omega n}$ (2)

comparing (1) and (2) $\Rightarrow$ $X(e^{j\omega}) = X_p(j\Omega)\Big|_{\Omega = \frac{\omega}{T_s}}$

periodic : 2π



d) similarly $X_d(e^{j\omega}) = X_{pd}(j\Omega)\Big|_{\Omega = \frac{\omega}{T_{sd}}}$



e) Yes.

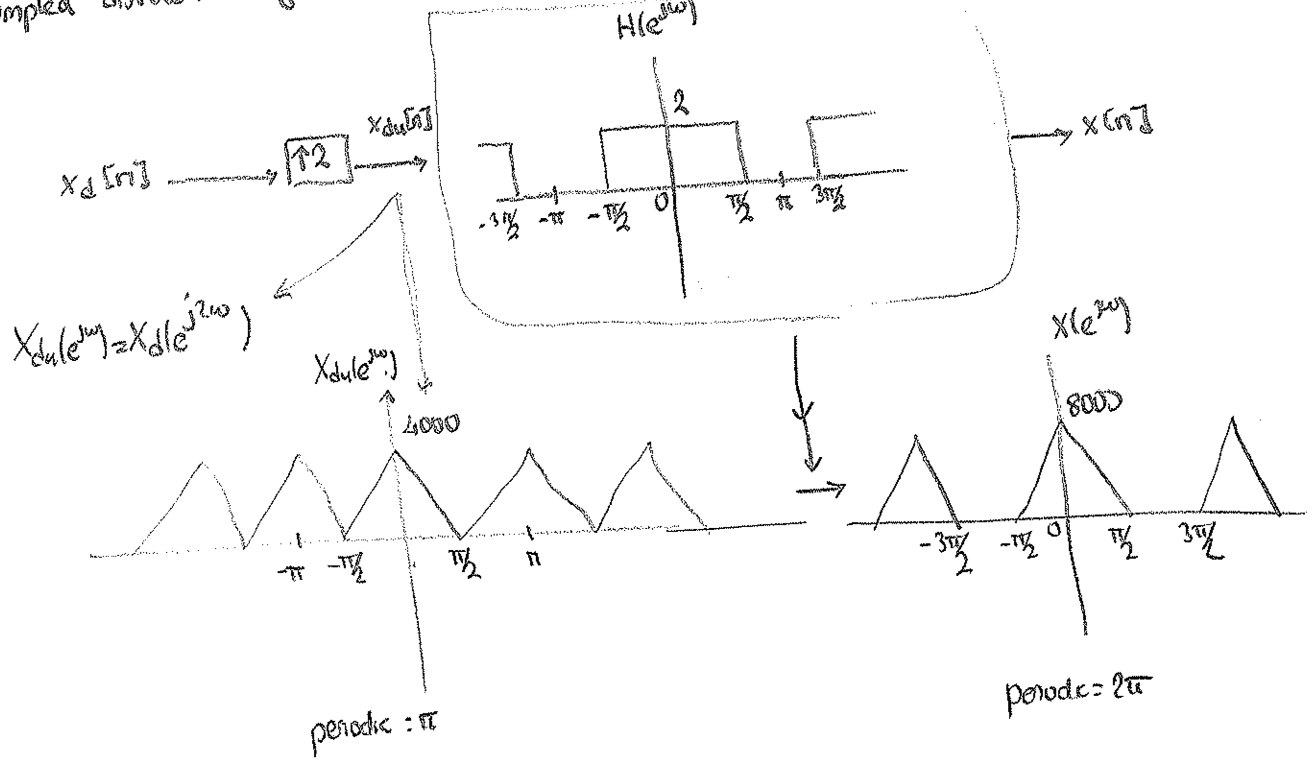
$x[n] \longrightarrow \boxed{\downarrow 2} \longrightarrow x_d[n]$ (just downsample by 2)

$$X_d(e^{j\omega}) = \frac{1}{2}\sum_{k=0}^{1} X(e^{j\frac{(\omega-2\pi k)}{m}}) = \frac{1}{2}X(e^{j\frac{\omega}{2}}) + \frac{1}{2}X(e^{j\frac{(\omega-2\pi)}{2}})$$

$$= \frac{1}{2}X(e^{j\omega/2}) + \frac{1}{2}X(-e^{j\omega/2})$$

Output of downsampler : $x[2n] = x_a(n2T_s) = x_a(n.T_{sd}) = x_d[n]$

f) Yes. Compare the graphs of $X_d(e^{j\omega})$ and $X(e^{j\omega})$: $X_d(e^{j\omega})$ contains all the information that $X(e^{j\omega})$ has. (but not true in general. If the continuous signal is critically sampled and then the discrete time signal is downsampled; you cannot reconstruct the original discrete-time signal from the downsampled discrete-time signal.)



Q2. $x_c(t) = \cos 2\pi 450 t$

$f_s = 1.5\ kHz$

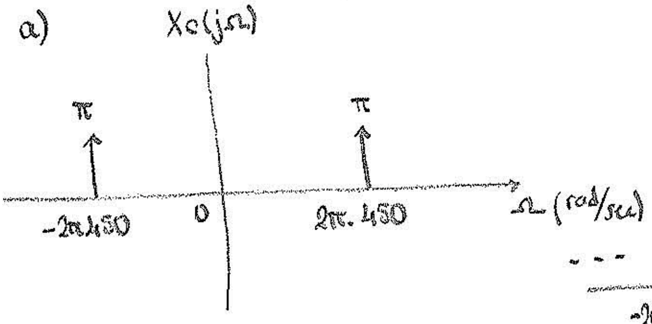
$x[n] = x_c(nT_s)$

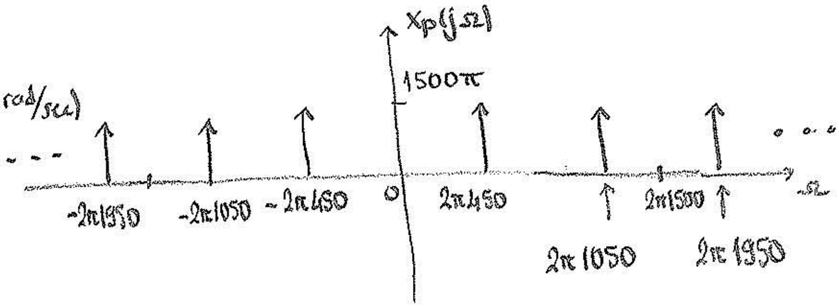
512 samples of $x[n]$ are present.

let $x_p(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta\left(t - n \cdot \frac{1}{1500}\right)$

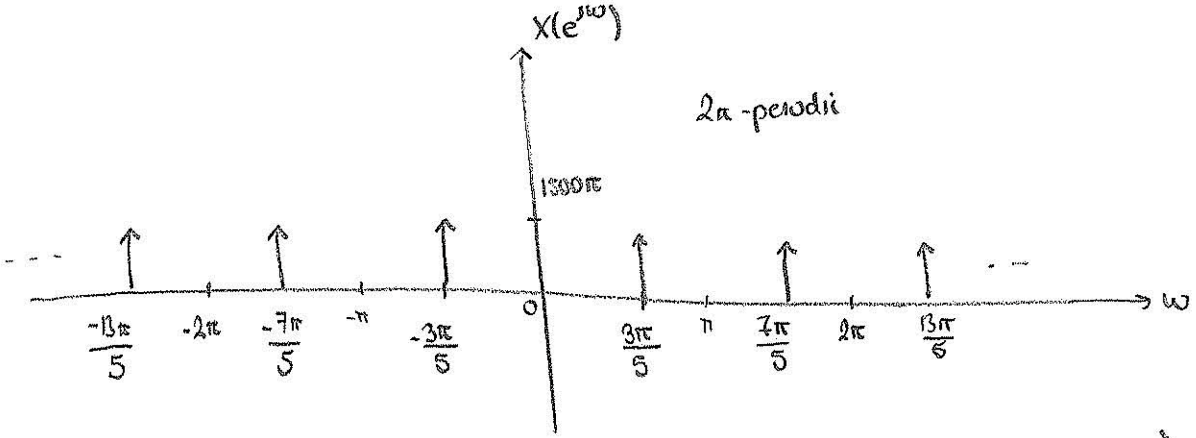
$X_p(j\Omega) = 1500 \sum_{n=-\infty}^{\infty} X_c(j(\Omega - k \cdot 2\pi \cdot 1500))$

a)





$X(e^{j\omega}) = X_p(j\Omega)\Big|_{\Omega = \frac{\omega}{T_s} = 1500\omega}$



b) $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} \cdot k \cdot n}$ for $k = 0, 1, \ldots, 511$ (512-point DFT of $x[n]$)

compare $X[k]$ and $X(e^{j\omega})$
assuming $x[n]$ periodic with N $\Bigg\} \Rightarrow \omega = \frac{2\pi}{N} \cdot k$

$\frac{3\pi}{5} = \frac{2\pi}{512} \cdot k_1 \Rightarrow k_1 = 153{,}6$

$\frac{7\pi}{5} = \frac{2\pi}{512} \cdot k_2 \Rightarrow k_2 = 358{,}4$