Discrete-time Signals and Systems
Final Exam
Fall Semester 2006

Duration: 110 minutes

Attempt all questions and **show your work. If you don't show your work you get zero credit.**

Q0 Let the sampling frequency be $f_s = 10\,KHz$

Actual frequency is $f_0 = 2\,KHz$. What is the normalized angular frequency ω_0 corresponding to $f_0 = 2\,KHz$?

Answer: $\omega_0 = 0.2\pi$

1 Consider the Butterworth analog filter

20 pts

$$|H(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^2}$$

(a) Design a discrete-time IIR low-pass filter with cut-off $\omega_c = \pi/2$, and

(b) Design a discrete-time IIR ~~low~~ high-pass filter with cut-off $\omega_c = \pi/2$, using the above analog prototype filter.

Q2
10 pts

Consider the input signal

$$x[n] = \{ \ldots, 1, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, 1, 2, 2, 2, 2, \ldots \}$$

1 2 3 4 5 6 7

a) Filter this signal using the low-pass filte[r] that you designed in Question 1.

b) Filter this signal using the high-pass filter that you designed in Question 1.

c) Comment on filter outputs.

d) If you are not sure about your design in Question 1 you can use any low-pass and high-pass filter pair you like, but you will get only 5 points!

LP filter makes a

3) 11

Consider an AR(1) random process, $x[n]$ genereted using a recursive manner as follow

$$x[n] = \alpha\, x[n-1] + u[n]$$

where $u[n]$ is a white random process with mean $=0$ and variance $\sigma_u^2 = 1$.

a) What is $r_u[n]$? What is $r_x[0]$, $r_x[1]$ and $r_x[2$

b) L.M.M.S.E (Linear minimum mean square err optimal predictor for this random process obtained using the "orthogonality principle". Wh is the orthogonality principle?

c) The L.M.M.S.E predictor is $\hat{x}[n] = 0.8\, x[n-1$
What should be the value of the parameter α
Use the orthogonality principle to determine α.

d) Let $\tilde{x}[n] = a_1\, x[n-1] + a_2\, x[n-2]$. Determine a_1 and a_2 using the orthogonality principle.

e) What are the impulse responses of the predictors in parts (c) and (d).

$r_u[1] = ?$

Q4 (a) Describe an N-point DFT using two $\frac{N}{2}$-point DFT's (N is divisible by 2).

(b) Let $X(e^{j\omega}) = \frac{1}{2} + \frac{1}{2}\cos\omega$. $X[k]$ is defined as

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{64}}, \quad k=0,1,\dots,63.$$

Find $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k n}{N}}, \quad n = 0,1,2,\dots,63 = N-1$

Hint: (a) You can use decimation-in-time or decimation-in-frequency methods.

(b) Parts (a) & (b) can be solved independently!

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Bilinear Transformation: $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$ $\quad H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$

Nth Order All Pass Filter: $G(z) = \prod_{i=1}^{N} \frac{z^{-1} - \alpha_i^*}{1 - \alpha_i z^{-1}}$

Mean Estimation: $\hat{m}_x = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$ or $\hat{m}_x = \sum_{x=x_i} xP(x)$

Variance Estimation: $\hat{\sigma}_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x[n] - \hat{m}_x)^2$ or $\hat{\sigma}_x^2 = \sum_{x=x_i}(x - \hat{m}_x)^2 P(x)$

Auto-Correlation Sequence and its Estimation:

$r_x(n_1,n_2) = E[x[n_1]x[n_2]]$ $n_1, n_2 \in Z$. With WSS assumption $\Rightarrow r_x(n) = r_x(|n|)$ where $n = n_1 - n_2$

$\hat{r}_x(n) = \frac{1}{N}\sum_{i=0}^{N-1-n} x[i]x[i+n]$ $\quad \hat{r}_x(n) = \hat{r}_x(-n)$ where $\hat{r}_x(n)$ is estimation for $r_x(n)$

Auto-Covariance Sequence and its Estimation:

$c_x(n_1,n_2) = E[(x[n_1] - m_{n_1})(x[n_2] - m_{n_2})] = r_x(n_1,n_2) - m_{n_1}m_{n_2}$; $\hat{c}_x(n) = \hat{r}_x(n) - \hat{m}_x^2$; $\hat{c}_x(0) = \hat{\sigma}_x^2$

Auto-Correlation Normal Equations (ACNE) for 2nd Degree:

$\mathbf{R}_x = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}$; $\bar{\mathbf{r}}_x = \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$; $\mathbf{R}_x\bar{\mathbf{a}} = \bar{\mathbf{r}}_x$; $\bar{\mathbf{a}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$; $\hat{x}[n] = a_1 x[n-1] + a_2 x[n-2]$

Butterworth Filter Design:

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}} \qquad \begin{array}{ll} 1 \ge |H_a(j\Omega)| \ge M_P & for\ 0 \le \Omega \le \Omega_P \\ |H_a(j\Omega)| \le M_S & for\ \Omega \ge \Omega_S \end{array} \qquad N_{exact} = \frac{\log\left(\frac{\frac{1}{M_S^2} - 1}{\frac{1}{M_P^2} - 1}\right)}{2\log\left(\frac{\Omega_S}{\Omega_P}\right)}$$

$$N = \lceil N_{exact} \rceil \qquad \Omega_C = \frac{\sqrt[N]{M_P}}{\sqrt[2N]{1 - M_P^2}}\Omega_P \qquad H_a(s) = \prod_{k=0}^{N-1} \frac{|s_k|}{s - s_k} \qquad s_k = \Omega_C e^{j\frac{\pi}{2}} e^{j\frac{2k+1}{2N}\pi}$$

Table 6.1: Some commonly used z-transform pairs.

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n \mu[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n \mu[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(n+1)\alpha^n \mu[n]$	$\frac{1}{(1 - \alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_o n)\mu[n]$	$\frac{1 - (r\cos\omega_o)z^{-1}}{1 - (2r\cos\omega_o)z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n \sin\omega_o n)\mu[n]$	$\frac{(r\sin\omega_o)z^{-1}}{1 - (2r\cos\omega_o)z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$