1) Given $x_a(t)$. We want to sample this signal. Assume that you have an A/D converter with a high sampling rate. How do you determine an efficient sampling frequency for $x_a(t)$?

If we know the highest frequency that contained in $x_a(t)$ we can choose our sampling rate as it is greater than the 2 times of the highest frequency in the $x_a(t)$. (Nyquist criterion)

Q2 Given $x[n] = \{\frac{1}{4}, \frac{1}{2}, \underset{\substack{\uparrow \\ n=0}}{1}, \frac{1}{2}, \frac{1}{4}\}$.

a) Compute the 8-point DFT of $x[n]$.

b) What is the relation between $X(e^{jw})$ and the 8-point DFT, $X[k]$? (Find $X(e^{jw})$).

c) Let $Y[k] = [\underset{\substack{\uparrow \\ k=0}}{1}, 1, 1, 1, 1, \ldots\ldots, \underset{\substack{\uparrow \\ k=1023}}{1}]$. Find $y[n]$!

d) Let $y[n] = [\underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, 1, \ldots\ldots\ldots, \underset{\substack{\uparrow \\ n=1023}}{1}]$. Find $Y[k]$.

e) Prove that $X[k] = X^*[N-k]$ when x is real.

a) $x_1[n] = \{\underset{\uparrow}{1}, 1/2, 1/4, 0, 0, 0, 1/4, 1/2\}$, $X[k] = X_1[k]$

$$X[k] = \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{7} x_1[n] e^{-j\frac{\pi k n}{4}} \quad \text{for } k = 0, 1, \ldots, 7$$

$$= 1 + \frac{1}{2} e^{-j\frac{\pi k}{4}} + \frac{1}{4} e^{-j\frac{\pi k}{2}} + \frac{1}{4} e^{-j\frac{3\pi}{2}k} + \frac{1}{2} e^{-j\frac{7\pi k}{4}}$$

$$= 1 + \cos\left(\frac{\pi k}{4}\right) + \frac{1}{2}\cos\left(\frac{\pi k}{2}\right)$$

$X[0] = \frac{5}{2}$, $X[1] = 1 + \frac{\sqrt{2}}{2}$, $X[2] = 1/2$, $X[3] = 1 - \frac{\sqrt{2}}{2}$

$X[4] = 1/2$, $X[5] = 1 - \frac{\sqrt{2}}{2}$, $X[6] = \frac{1}{2}$, $X[7] = 1 + \frac{\sqrt{2}}{2}$

$$\Rightarrow X[k] = \{\underset{\substack{\uparrow \\ n=0}}{5/2}, 1 + \frac{\sqrt{2}}{2}, 1/2, 1 - \frac{\sqrt{2}}{2}, \frac{1}{2}, 1 - \frac{\sqrt{2}}{2}, 1/2, 1 + \frac{\sqrt{2}}{2}\}$$

S/S

b)
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = \frac{1}{4} e^{j2w} + \frac{1}{2} e^{jw} + 1 + \frac{1}{2} e^{-jw} + \frac{1}{4} e^{-j2w}$$

$$= \frac{1}{2}\cos(2w) + \cos(w) + 1$$

S/S

$$\Rightarrow X[k] = X(e^{jw})\Big|_{w = \frac{2\pi k}{8}} \quad \text{for } k = 0, 1, \ldots, 7$$

N=1024

(c) $$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k] e^{\frac{j2\pi kn}{N}} = \frac{1}{1024}\sum_{k=0}^{1023} e^{\frac{j2\pi kn}{1024}}$$

for $n=0$

$$y[n] = \frac{1024}{1024} = 1$$

for $n \neq 0$

$$y[n] = 0$$

$$\Rightarrow y[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 0, 0 \ldots\ldots, 0\}$$

4/4

N=1024

(d) $$Y[k] = \sum_{n=0}^{N-1} y[n] e^{\frac{j2\pi kn}{N}} = \sum_{n=0}^{1023} e^{\frac{j2\pi kn}{1024}}$$

for $n=0 \Rightarrow Y[k] = 1024$

for $n \neq 0 \Rightarrow Y[k] = 0 \quad \Rightarrow Y[k] = \{\underset{\substack{\uparrow \\ n=0}}{1024}, 0, 0, 0 \ldots\ldots, 0\}$

4/4

(e) $$X[N-k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi n(N-k)}{N}}$$

$$= \sum_{n=0}^{N-1} x[n] \underbrace{e^{-j\frac{2\pi nN}{N}}}_{1} e^{j\frac{2\pi nk}{N}} = \sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi nk}{N}}$$

$$\Rightarrow X^*[N-k] = \sum_{n=0}^{N-1} x^*[n] e^{-j\frac{2\pi nk}{N}} \underset{\substack{\uparrow \\ \text{since } x[n] \\ \text{is real}}}{=} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} = X[k]$$

4/4

3) Given the transfer function

$$H(z) = \frac{-0.8 + z^{-1}}{1 - 0.8 z^{-1}}$$ of a system.

a) Plot $|H(e^{j\omega})|$. What type of a filter is this?

b) Find the I/O relation corresponding to this filter.

c) Is this filter stable (i) when the recursion that you obtained in part (b) is implemented in a causal manner? (ii) Is it stable when the recursion is implemented in an anticausal manner?

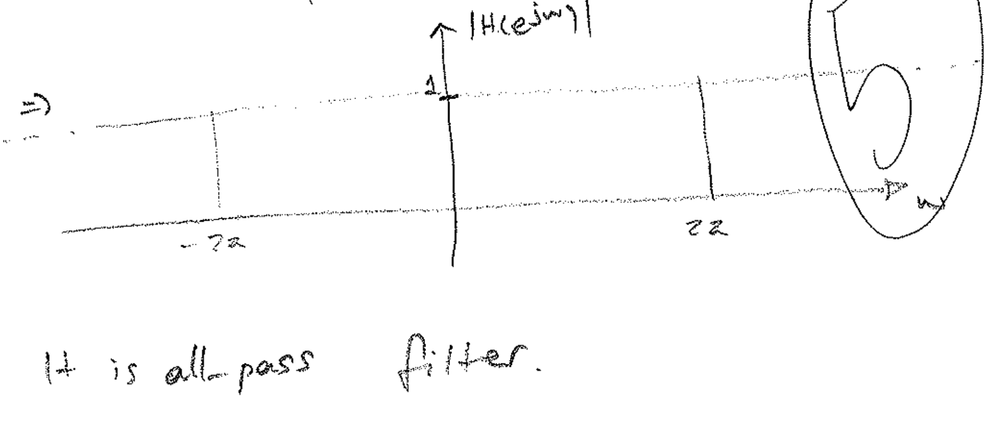
d) Let the input $x[n] = \{\ldots\ldots, 1, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, 2, 2, 2, 2, \ldots\}$

Find $y[n]$ using causal recursion (assume $y[-1]=0$)

e) Comment on the shape of output $y[n]$.

(a) $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \dfrac{-0.8 + e^{-j\omega}}{1 - 0.8 e^{-j\omega}} = e^{-j\omega} \cdot \dfrac{1 - 0.8 e^{j\omega}}{1 - 0.8 e^{-j\omega}}$

$|H(e^{j\omega})| = \underbrace{|e^{-j\omega}|}_{1} \left| \dfrac{1 - 0.8 e^{j\omega}}{1 - 0.8 e^{-j\omega}} \right| = \sqrt{\dfrac{(1 - 0.8\cos\omega)^2 + (\sin\omega)^2}{(1 - 0.8\cos\omega)^2 + (\sin\omega)^2}} = 1$



It is all-pass filter.

(b) $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{-0.8 + z^{-1}}{1 - 0.8z^{-1}}$

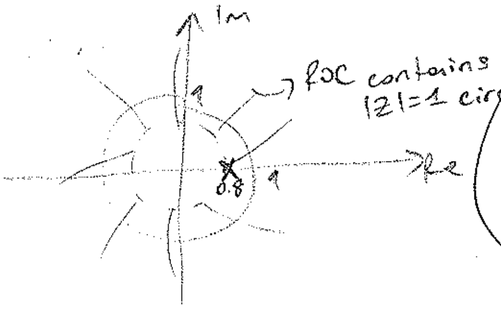
$\Rightarrow Y(z) = 0.8 z^{-1} Y(z) - 0.8 X(z) + z^{-1} X(z)$

$y[n] = 0.8\, y[n-1] - 0.8\, x[n] + x[n-1]$

5

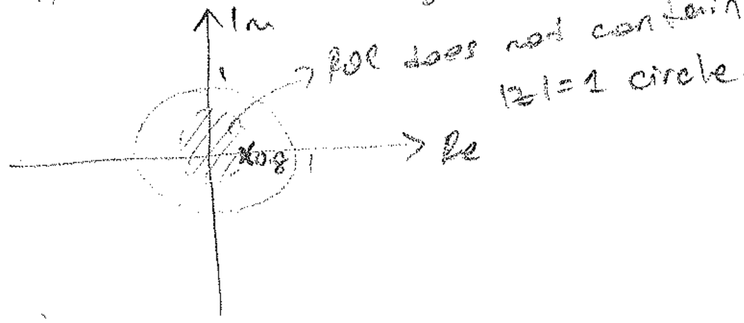
(c) There is one pole at $z = 0.8$. To get $|z| = 1$ circle in the ROC we need to use the causal one.

i) It is stable.



5

ii) It is not stable.



(d) $y[0] = 0.8\, y[-1] - 0.8\, x[0] + x[-1] = 0.2$

$y[1] = 0.8\, y[0] + 0.2 = 0.36$

$y[2] = 0.8\, y[1] + 0.2 = 0.36 \times 0.8 + 0.2 = 0.488$

$y[3] = 0.8\, y[2] - 0.6$

$y[4] = 0.8\, y[3] + 0.4$

$y[5] = 0.8\, y[4] + 0.4$

⋮

5

(e) It is always increasing after $y[4]$ and reaches infinity at $n \to \infty$

2

4) Consider the Butterworth analog prototype $\left|H_a(j\Omega)\right|^2 = \dfrac{1}{1+(j\Omega/j\Omega_c)^{2}}$

a) Design a discrete-time IIR low-pass filter with cut-off $\omega_c = \frac{\pi}{4}$.

b) Design a discrete-time IIR high-pass filter with cut-off $\omega_c = \frac{\pi}{4}$.

(a) when $N=1$ for Butterworth filter $H_a(s) = \dfrac{\Omega_c}{s+\Omega_c}$

By bilinear transformation

$$\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) = \frac{2}{T}\tan\left(\frac{\pi}{8}\right) = \frac{0.8284}{T}$$

15/15

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\Rightarrow H(z) = \frac{\Omega_c}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\Omega_c} = \frac{0.4142}{\frac{1-z^{-1}}{1+z^{-1}}+0.4142} = \frac{0.4142(1+z^{-1})}{1.4142-0.5858z^{-1}}$$

$$= \frac{0.4142}{1.4142}\left[\frac{1}{1-\frac{0.5858}{1.4142}z^{-1}} + \frac{z^{-1}}{1-\frac{0.5858}{1.4142}z^{-1}}\right]$$

$$\Rightarrow h_{lp}[n] = \frac{0.4142}{1.4142}\left(\frac{0.5858}{1.4142}\right)^{n}u[n] + \frac{0.4142}{1.4142}\left(\frac{0.5858}{1.4142}\right)^{n-1}u[n-1]$$

(b) $h_{hp}[n] = \delta[n] - h_{lp}[n] = \delta[n] - \dfrac{0.4142}{1.4142}\left[\left(\dfrac{0.5858}{1.4142}\right)^{n}u[n] + \left(\dfrac{0.5858}{1.4142}\right)^{n-1}u[n-1]\right]$

7/7

$$= \frac{1}{1.4142} - \frac{0.4142}{1.4142}\cdot\frac{1.5858}{1.41422}\left(\frac{0.5858}{1.4142}\right)^{n-1}u[n-1]$$

5\. Let $x[n] = 0.9\,x[n-1] + u[n]$, where $u[n]$ is white, zero-mean r.p. with variance 1.

a) Is $x[n]$ a Wide-sense stationary r.p.? Explain.

b) Determine the autocorrelation function of $x[n]$.

c) Determine the first order LMMSE predictor for $x[n]$. ($\hat{x}[n] = a\,x[n-1]$) Show your work!

d) Given a realization of $u[n] = \{\underset{\uparrow}{1}, 0, -1, 2, 0\}$. Obtain the corresponding realization of $x[n]$ (assume $x[-1] = 0$).

e) Estimate $x[5]$ using the predictor obtained in part (c) and from the predictor (obtained) from the data in (d). Compare the results of the two predictors.

Notice that you can still attempt parts (d) & (e) even if you cannot solve (a)–(c).