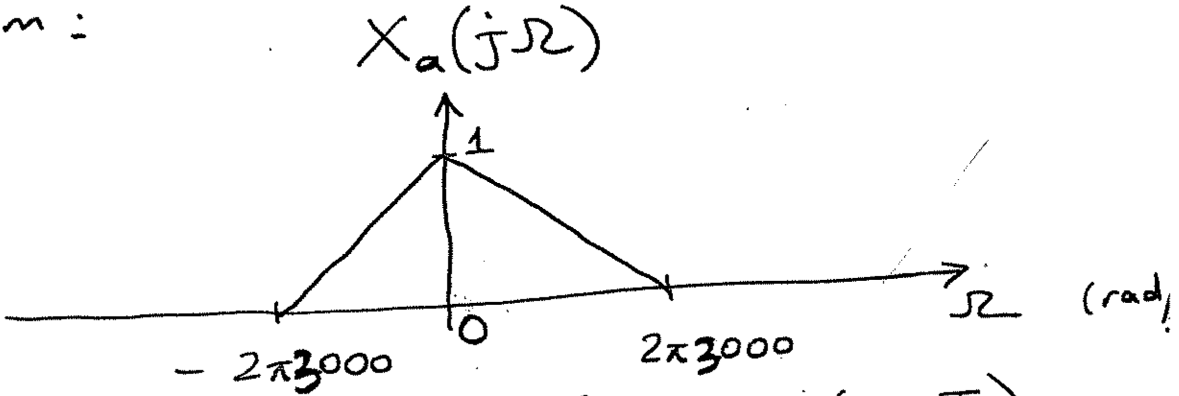
#1 || Given $x_a(t)$ with continuous Fourier Transform:



a)
$$\begin{tikzcd} x_a(t) \arrow[r] & \otimes \arrow[r] & x_p(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t-nT_s) \\ & \tilde{x}(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s) \arrow[u] & \end{tikzcd}$$

$f_s = \frac{1}{T_s} = 8$ KHz.

Plot $X_p(j\Omega)$.

b) Let the sampling frequency be $f_{sd} = 4$KH

Plot $X_{pd}(j\Omega)$
$$\left(\begin{tikzcd} x_a(t) \arrow[r] & \otimes \arrow[r] & x_{pd}(t) \\ & \sum_n \delta(t-nT_{sd}) \arrow[u] & \end{tikzcd} \right.$$
$, T_{sd} = \frac{1}{f_{sd}}$

c) Plot the D.T.F.T. of $x[n] = \{x_a(nT_s)\}_{n=-\infty}^{\infty}$

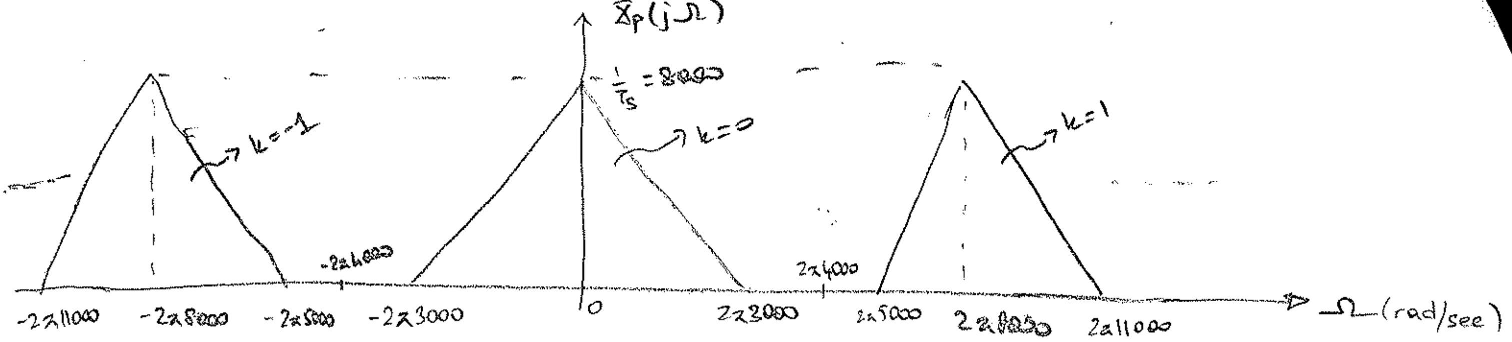
d) Plot the D.T.F.T. of $x_d[n] = \{x_a(nT_{sd})\}_{n=-\infty}^{\infty}$.

e) Can you obtain $x_d[n]$ from $x[n]$? If yes, draw the block-diagram of your system obtaining $x_d[n]$ from $x[n]$. If no, explain!

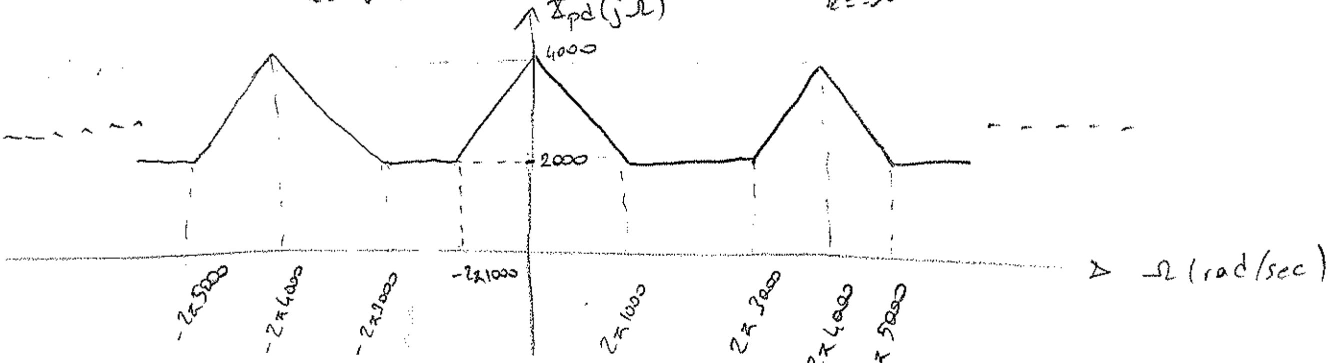
f) Can you obtain $x[n]$ from $x_d[n]$? If yes, draw the block-diagram of your system obtaining $x[n]$ from $x_d[n]$. If no, explain your answer!

ⓐ $X_p(j\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_s))$, $\Omega_s = \frac{2\pi}{T_s}$

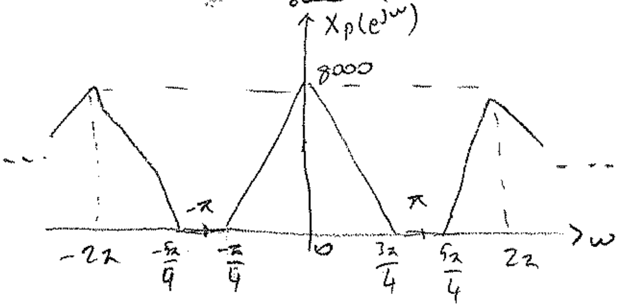
$\Rightarrow X_p(j\Omega) = 8000 \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k2\pi \cdot 8000))$



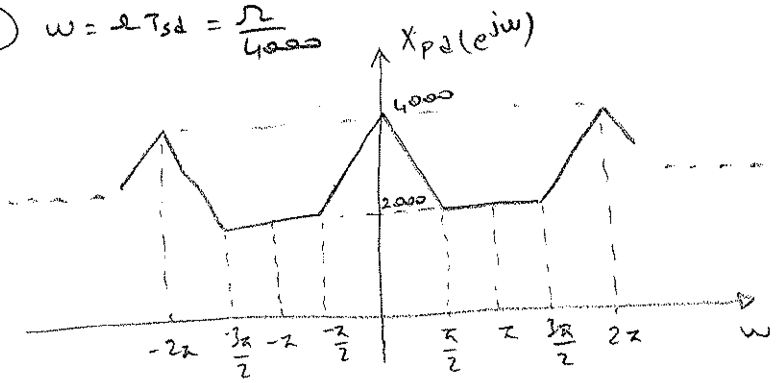
ⓑ $X_{pd}(j\Omega) = \frac{1}{T_{sd}} \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_{sd})) = 4000 \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k2\pi 4000))$



ⓒ $w = \Omega T_s = \frac{\Omega}{8000}$



ⓓ $w = \Omega T_{sd} = \frac{\Omega}{4000}$



ⓔ yes

$X[n] \rightarrow \boxed{2\downarrow} \rightarrow X_d[n]$

ⓕ NO

Because $f_{sd} \overset{?}{>} 2 \cdot 3000 = 6000$ ✗ ⟶ Therfore there is aliasing ($X_{pd}(j\Omega)$ also shows aliasing). Therefore we cannot obtain $X[n]$ from $X_d[n]$.

Due to aliasing we corrupted our original signal's frequency spectrum.

2

$\frac{1}{2}, 1, \frac{1}{2}$

Let $x[n] = \{\dots, 1, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 2, 2, 2, 1\}$ and given the low-pass filter $h[n] = \{\frac{1}{4}, \underset{\uparrow}{\frac{1}{2}}, \frac{1}{4}\}$

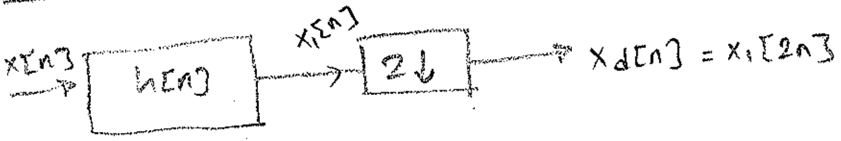
a) Decimate $x[n]$ by a factor of 2 using the above low-pass filter.

b) Interpolate $x[n]$ by a factor of 2 using the same low-pass filter.

c) Plot the frequency response of $h[n]$.

ⓐ Decimation by factor 2

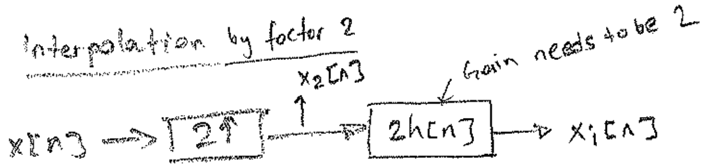
$\frac{3}{2}$ $\frac{5}{4}$



$x_1[n] = x[n] * h[n] = \{\dots, 1, 1, \underset{\substack{\uparrow \\ n=0}}{1}, \frac{5}{4}, \frac{7}{4}, 2, \frac{7}{4}, 1, \frac{1}{4}\}$

$x_d[n] = x_1[2n] = \boxed{\{\dots\dots 1, \underset{\substack{\uparrow \\ n=0}}{1}, \frac{7}{4}, \frac{7}{4}, \frac{1}{4}\}}$

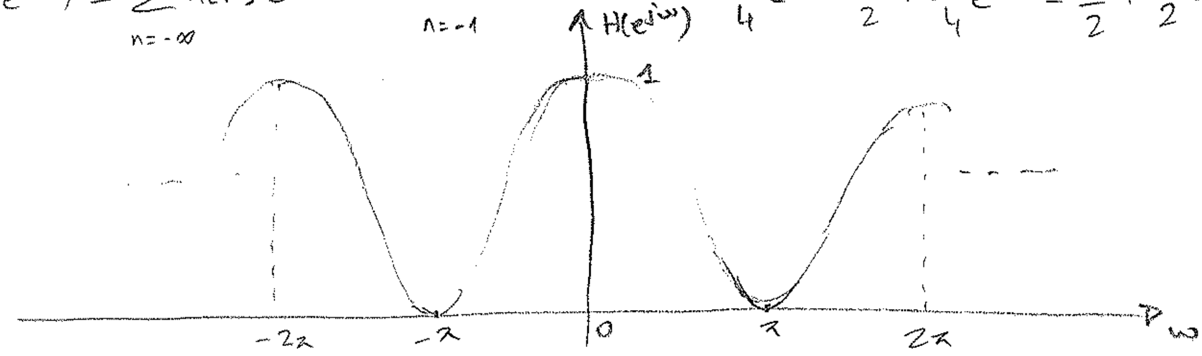
ⓑ Interpolation by factor 2



$x_2[n] = \begin{cases} X[n/2] & , n \text{ even} \\ 0 & , n \text{ odd} \end{cases} \Rightarrow x_2[n] = \{\dots, 0, 1, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 0, 1, 0, 2, 0, 2, 0, 2, 0, 1\}$

$x_i[n] = x_2[n] * (2h[n]) = \boxed{\{\dots\dots, 1, 1, 1, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, \frac{3}{2}, 2, 2, 2, 2, 2, \frac{3}{2}, 1, \frac{1}{2}\}}$

ⓒ $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=-1}^{n=1} h[n] e^{-j\omega n} = \frac{1}{4} e^{j\omega} + \frac{1}{2} + \frac{1}{4} e^{-j\omega} = \frac{1}{2} + \frac{1}{2}\cos(\omega)$

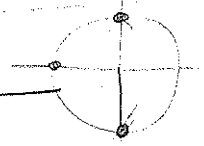


3) Given $x_1[n] = \{\underset{\substack{\uparrow\\ n=0}}{1}, 1, 0, 0\}$ and $x_2[n] = \{\underset{\substack{\uparrow\\ n=0}}{1}, 1, 1, 1\}$

a) Compute the 4-point DFT's of x_1 and x_2.

b) Compute the 4-point circular convolution of x_1 and x_2 using DFT.

c) What should be the size of DFT such that the circular convolution produces the actual convolution result



a) DFT-4

$$X_1[k] = \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{3} x_1[n] e^{-j\frac{2\pi kn}{4}} = 1 + e^{-j\frac{\pi k}{2}} \quad , \text{for } k=0,1,2,3$$

$\Rightarrow X_1[0] = 2$

$X_1[1] = 1-j$

$X_1[2] = 0$

$X_1[3] = 1+j$

✓

10|10

$$X_2[k] = \sum_{n=0}^{3} x_2[n] e^{-j\frac{2\pi kn}{4}} = 1 + e^{-j\frac{\pi k}{2}} + e^{-j\pi k} + e^{-j\frac{3\pi}{2}k} = 1 + (-j)^k + (-1)^k + (j)^k \quad , \text{for } k=0,1,2,3$$

$\Rightarrow X_2[0] = 4$

$X_2[1] = 0$

$X_2[2] = 2$

$X_2[3] = 0$

✓

b) $y[n] = x_1[n] \textcircled{4} x_2[n] \Rightarrow DFT_4\{y[n]\} = DFT_4\{x_1[n]\} \cdot DFT_4\{x_2[n]\}$

DFT-4 of $y[n]$

$Y[k] = X_1[k] X_2[k]$, for $k=0,1,2,3$

$\Rightarrow Y[0] = 8$

$Y[1] = 0$

$Y[2] = 0$

$Y[3] = 0$

$$y[n] = IDFT_4(Y[k]) = \frac{1}{4}\sum_{k=0}^{3} Y[k] e^{j\frac{2\pi kn}{4}} = 2 \quad , \text{for } n=0,1,2,3$$

$\boxed{y[n] = 2 \quad n=0,1,2,3} \rightarrow$ 4-poin circular conv of x_1 and x_2

10|10

ⓒ DFT size needs to be $2+4-1=5$

Size of $x_2[n]$ is 4.

Size of $x_1[n]$ is 4 but last 2 element of $x_1[n]$ is 2 zeros and since we will padd zeros to x_1 and x_2 to compute actual convolution by circular convolution size of x_1 can be considered as $\underline{\underline{2}}$.

Since we are looking minimum size

4^4

$\frac{1}{2} - j\frac{\sqrt{3}}{2}$

Q4 a) Draw the flow-graph of N=6-point decimation-in-time fast Fourier Transform (FFT) algorithm.

$e^{-j\frac{2\pi k}{6}}$ $e^{j\frac{2\pi k}{6}}$ $1 + W_6^k \frac{1}{2} + W_6^{5k}\frac{1}{2}$

$1 + \cos\left(\frac{2\pi k}{6}\right)$

b) Let $x[n] = \left\{\frac{1}{2}, \underset{\substack{\uparrow \\ n=0}}{1}, \frac{1}{2}\right\}$. Calculate the N=6 point DFT of $x[n]$ using the flowgraph in (a). (If you are not sure about your flowgraph calculate $X[k]$ using the definition of DFT but you will get a half of the full credit)

ⓐ Step 1

$$\overset{\text{DFT-6}}{X[k]} = \sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi k n}{6}} \quad , \; W_N^{kn} = e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0,2,4} x[n] e^{-j\frac{2\pi k n}{6}} + \sum_{n=1,3,5} x[n] e^{-j\frac{2\pi k n}{6}}$$

$$= \sum_{\substack{m=0,1,2 \\ \uparrow \\ n=2m}} x[2m] e^{-j\frac{2\pi k m}{3}} + \sum_{\substack{m=0,1,2 \\ \uparrow \\ n=2m+1}} x[2m+1] e^{-j\frac{2\pi k m}{3}} e^{-j\frac{2\pi k}{6}}$$

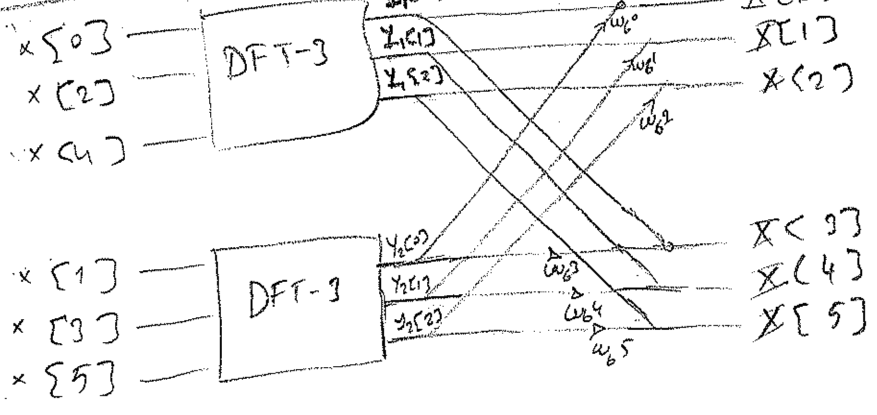
$$= DFT_3(x[2m]) + W_6^k DFT_3(x[2m+1])$$

Step 2

$$\overset{\text{DFT-3}}{Y[k]} = \sum_{n=0}^{2} y[n]\, e^{-j\frac{2\pi k n}{3}} = y[0] + y[1] e^{-j\frac{2\pi k}{3}} + y[2] e^{-j\frac{4\pi k}{3}}$$

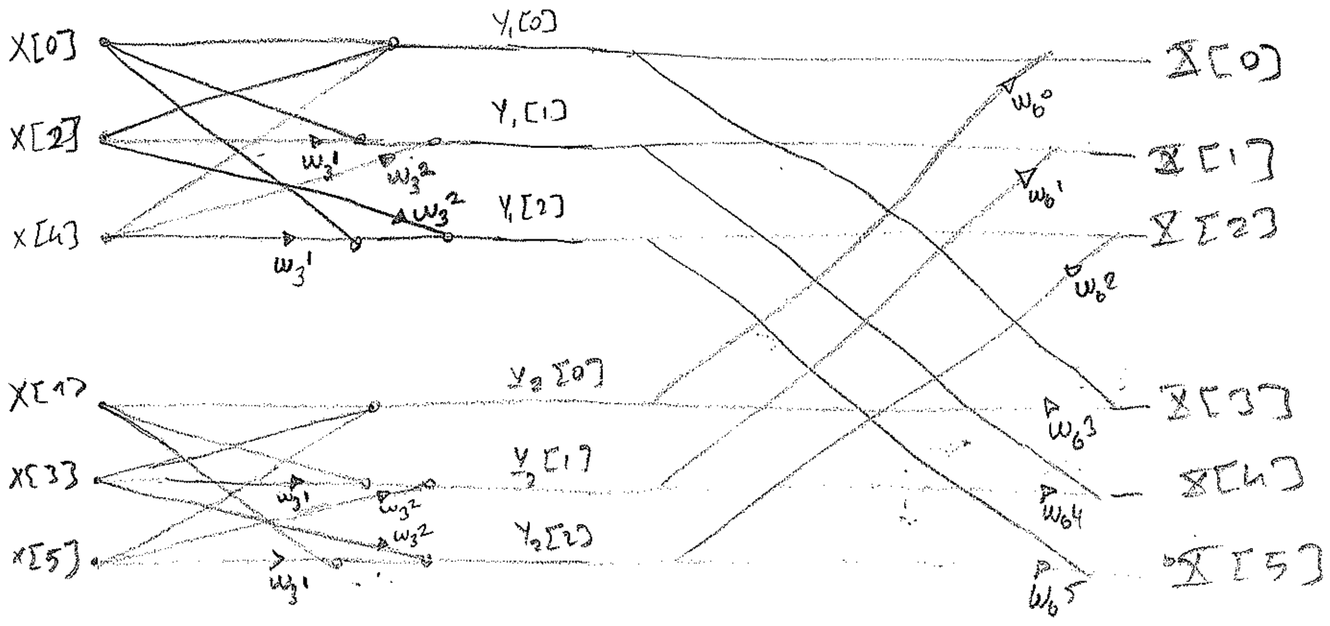
$$= y[0] + y[1] W_3^k + y[2] W_3^{2k}$$

Step 1 :



⟹

Step 1 + Step 2



ⓑ $\tilde{x}[n] = \{ \underset{\overset{\uparrow}{n=0}}{1}, \frac{1}{2}, 0, 0, 0, \frac{1}{2} \}$

$\mathbb{X}[k] = DFT_6\{x[n]\} = DFT_6\{\tilde{x}[n]\}$

$\Rightarrow \quad Y_1[0] = x[0] = 1 \qquad , \; Y_2[0] = x[1] + x[5] = \frac{1}{2} + \frac{1}{2} = 1$

$Y_1[1] = x[0] = 1 \qquad Y_2[1] = x[1] + x[5] W_3^2 = \frac{1}{2} + \frac{1}{2} e^{-j\frac{4\pi}{3}}$

$Y_1[2] = x[0] = 1 \qquad Y_2[2] = x[1] + x[5] W_3^1 = \frac{1}{2} + \frac{1}{2} e^{-j\frac{2\pi}{3}}$

$\mathbb{X}[0] = Y_1[0] + Y_2[0] W_6^0 = 1 + 1 = 2$

$\mathbb{X}[1] = Y_1[1] + Y_2[1] W_6^1 = 1 + \left(\frac{1}{2} + \frac{1}{2} e^{-j\frac{4\pi}{3}}\right) e^{-j\frac{2\pi}{6}} = 1 + \frac{1}{2} e^{-j\frac{2\pi}{6}} + \frac{1}{2} e^{j\frac{2\pi}{6}} = 1 + \cos\left(\frac{2\pi}{6}\right) = \boxed{\frac{3}{2}}$

$\mathbb{X}[2] = Y_1[2] + Y_2[2] W_6^2 = 1 + \left(\frac{1}{2} + \frac{1}{2} e^{-j\frac{2\pi}{3}}\right) e^{-j\frac{2\pi}{3}} = 1 + \frac{1}{2} e^{-j\frac{2\pi}{3}} + e^{j\frac{2\pi}{3}} = 1 + \cos\left(\frac{2\pi}{3}\right) = \boxed{\frac{1}{2}}$

$\mathbb{X}[3] = Y_2[0] W_6^3 + Y_1[0] = 1 \cdot (-1) + 1 \boxed{= 0}$

$\mathbb{X}[4] = Y_2[1] W_6^4 + Y_2[1] = \left(\frac{1}{2} + \frac{1}{2} e^{-j\frac{4\pi}{3}}\right) e^{-j\frac{4\pi}{3}} + 1 = 1 + \frac{1}{2} e^{-j\frac{4\pi}{3}} + \frac{1}{2} e^{-j\frac{4\pi}{3}} = 1 + \cos\left(\frac{4\pi}{3}\right) = \boxed{\frac{1}{2}}$

$\mathbb{X}[5] = Y_2[2] W_6^5 + Y_1[2] = \left(\frac{1}{2} + \frac{1}{2} e^{-j\frac{2\pi}{3}}\right) e^{-j\frac{10\pi}{6}} + 1 = 1 + \frac{1}{2} e^{-j\frac{10\pi}{6}} + \frac{1}{2} e^{j\frac{10\pi}{6}} = 1 + \cos\left(\frac{5\pi}{3}\right) = \boxed{\frac{3}{2}}$

10/10

$\Rightarrow \mathbb{X}[k] = \{ \underset{\overset{\uparrow}{k=0}}{2}, \frac{3}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2} \}$