EXERCISES

A) Z–Transforms

1. Compute the z-transforms of the following sequences.

   a) $f_1(k) = \{1, 0, -1, 0, 1, 0, -1, 0, ...\}$

   b) $f_2(k) = \{0, 0, c^2, 0, c^4, 0, c^6, 0, ...\}$

   c) $f_3(k) = \begin{cases} \alpha, & k = 4m \\ \beta, & k = 4m + 1 \\ 0, & \text{otherwise} \end{cases}$

   d) $f_4(k) = (0.1)^k 1(k) + 0.5k(0.1)^k 1(k-1)$

   e) $f_5(k) = \{1, 0, 1, 2, 1, 0, 1, 2, ...\}$

2. Find the inverse z-transforms of

   a) $F_1(z) = \frac{z^6+2z^5-z^3+2z^2}{z^6-1}$

   b) $F_2(z) = \frac{z^2-0.25z}{z^2-0.5z+0.25}$

   c) $F_3(z) = \frac{z(z-1)}{z^2-1.25z+0.25}$

   d) $F_4(z) = \frac{z^3+z^2+z}{z^3+z^2+z+1}$

3. Find the inverse z-transform of $F(z) = z/(z - re^{j\theta})(z - re^{-j\theta})$, and plot $f(k)$ for $r = 0.5, 0.8,\ \theta = 0, \pi/3, 2\pi/3$.

4. Show that if $\{f(k)\}$ is a periodic sequence with period K, then $F(z)$ is of the form $F(z) = zp(z)/(z^K - 1)$, where $p(z)$ is a polynomial of degree $K - 1$.

B) Discrete Systems

1. The difference equation

$$y(k+2) = y(k+1) + y(k), k \geq 0$$

generates the Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, ...\}$ from the initial conditions $y(0) = 0y(1) = 1$.

   a) Obtain a state space description of the system.

   b) Find $\lim_{k\to\infty}\{y(k+1)/y(k)\}$.

2. Find the response of the system

$$y(k+2) - 0.1y(k+1) - 0.2y(k) = 1(k+1) + 1(k), \quad y(0) = y(1) = 0$$

   a) by using the z-transform,

   b) through a state–space description.

3. Given a discrete-time system described by the difference equation

$$y(k) = (1/6)y(k-1) + (1/6)y(k-2) + u(k) - u(k-1) + u(k-2)$$

   a) Obtain the transfer function.

   b) Obtain a state-space description.

   c) Find the weighting sequence.

   d) Find a finite duration input to have $y(k) = (1/2)^k$ for $k \geq 2$.

   e) Find an input $u(k) = \cos\omega k$ which results in $y_{ss}(k) = 0$.

   f) Compute the steady state response to input $u(k) = \sin(\pi k/2)$.

4. The forward path transfer function of a unity feedback discrete–time system is given as $G(z) = (0.1z + 0.74)/(z-1)(z+0.1)$. Find the output sequence $y(k)$ for a step input $r(k) = 1(k)$.

5. Given a system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & 1/4 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{aligned}$$

   a) Find a finite duration input which results in a zero-state response of the form $y(k) = (1/2)^k, k \geq 2$.

b) Compute the steady state response to inputs
i) $u(k) = \cos(\pi k)$ ii) $u(k) = \sin(\pi k/2)$.

6. Apply Jury's test to determine if the following characteristic equations have any roots outside the unit circle.

   a) $z^3 - 1.1z^2 + 0.01z + 0.405$

   b) $z^3 - 3.6z^2 + 4z - 1.6$

7. Consider the system in Q.9 with $u(k) = 0$.

   a) Check the stability of the zero input solution.

   b) Find the exact degree of exponential stability.

   c) Find a quadratic Liapunov function by solving the discrete Liapunov equation with $G = I$, and use it to compute an estimate of the degree of exponential stability.

C) Sampling, Sampled–Data Systems, Modified Z–Transform

1. Let $f(t)$ be a real signal, band-limited to $(4 \text{ KHz}, 5 \text{ KHz})$. Show that $f(t)$ can be reconstructed exactly from samples taken at a rate $f_s = 2$ KHz, and obtain an expression for $f(t)$ in terms of the samples $\{f(kT)\}, T = 1/f_s = 0.5$ msec.

2. In the closed–loop system in Fig.1, $T = 1, D(z) = 1, G(s) = 1/s, F(s) = 1$ and $r(t) = 1(t)$. Find by inspection and plot $e(t), u(t)$ and $y(t)$ for $0 \leq t \leq 2$. Find also $y(\infty)$.

Figure 1. Closed–loop sampled–data system

3. In the closed-loop system in Fig. 1, $T = 0.1, D(z) = 1, G(s) = 1/(s+1)$ and $F(s) = e^{-0.1s}$. Find the closed-loop pulse transfer function $H_T(z)$, and $\{y(kT)\}$ for $r(t) = 1(t)$.

4. In the closed–loop system in Fig.1, $D(z) = K, G(s) = 1/s$ and $F(s) = 1$.

   a) Determine the range of K in terms of T for closed–loop stability.

   b) Let $r(t) = 1(t)$. Find K to have $e(kT) = 0$ for $k \geq 1$.

   c) For the value of K found in (b), calculate and plot $y(t)$ for $0 \leq t \leq 4T$.

5. In the closed–loop system in Fig.1, $D(z) = K, G(s) = (1 - s)/s(1 + s)$ and $F(s) = 1$.

   a) Plot the root–locus for $T = 0.1$ and determine the maximum value of K for closed–loop stability.

   b) Repeat (a) for $T = 2$.

   c) Repeat (a) for the corresponding continuous–time system and compare the results.

6. For a signal $f(t)$ it is given that

$$F_T(z) = -z/(z^2 + 1), \quad F_{T,1/3}(z) = z^2/(z^2 + 1), \quad F_{T,2/3}(z) = z/(z^2 + 1)$$

   Find a closed–form expression for the sequence $f(kT/3)$.

7. In the following multirate system obtain an expression for $Y_T(z)$ in terms of the regular and modified z-transforms of $u(t)$ and $G(s)$.

8. In the following system $r(t) = t1(t)$ and $G(s) = 1/(s+1)$. Find $\{y(kT)\}$ for a) $\tau = T/2$, b)$\tau = 2T$.

D) Frequency Response Design

1. The simplified pulse transfer function of a single-axis satellite which relates the angular position of the satellite with respect to a fixed inertial axis to normalized input torque is given as $G(z) = K(z+1)/(z-1)^2$.

    a) Sketch the root-locus of the closed-loop system with unity feedback. Determine the range of K for which the closed-loop system is stable.

    b) Add a lead compensator to have dominant closed loop poles with $\zeta = 0.5$ and $\theta = \pi/4$. Sketch the root-locus of the compensated closed-loop system.

    c) Calculate the closed-loop transfer function of the compensated system, and use MATLAB to plot the step response.

2. The simplified transfer function of an antenna which relates the azimuth angle to normalized input torque is given as $G(s) = 1/s(1+10s)$. The system is to be controlled to have $K_v = 1$ and $M_p \leq 16\%, t_s \leq 10$ sec (which correspond to $\zeta = 0.5, \omega_n = 1$ rad/sec.)

    a) Design an analog controller to meet the specs without taking into account the effect of ZOH. Discretize the controller using pole-zero matching with $T = 0.1$ sec. Find the closed-loop pulse transfer function, and calculate the ζ and ω_n the actual closed-loop poles correspond to. Use MATLAB to plot the closed-loop step response. Find actual M_p and t_s, and compare with what you would expect from the closed-loop poles.

    b) Repeat (a) with $T = 0.5$, and by taking into account the effect of ZOH. (With $T = 0.5$, the phase lag due to ZOH is too large to ignore).

    c) Repeat (b) with a digital controller designed by the root-locus method for the open-loop pulse transfer function corresponding to $T = 0.5$.

3. The dynamics of a disk drive tracker head can be described by a simplified transfer function $G(s) = 1/s^2(s+1000)$. Design a digital controller for $T = 1$ msec, $t_s = 20$ msec, $M_p \leq 20\%$ by using

    a) s-plane frequency response design and emulation,

    b) w-plane frequency response design.

    In each case, find the step response of the closed-loop system, and t_s and M_p.

4. A system with $G(s) = 1/s^2$ is to be controlled digitally to have $K_v = 10$ sec^{-1}, $\zeta = 0.5$, and $w_n = 10$ rad/sec.

    a) Design an analog controller of the form $G_c(s) = Ks/(s+a)$ to meet the specs. (Neglect the effect of ZOH.)

    b) Discretize $G_c(s)$ using pole–zero matching with $T = 0.1$ sec. (Be careful in choosing the gain of the discrete controller. It should have the same low frequency behavior as $G_c(s)$.)

c) For the discrete controller in part (b), find the closed–loop pulse transfer function. Compute the actual values of K_v, ζ, w_n.

d) Design a discrete controller directly to have $\zeta = 0.5$, $w_n = 10$. What is K_v in this case?

5. In the closed–loop system in Fig.1, $T = 0.1, G(s) = 1/s(s+1)$ and $F(s) = 1$.

   a) Design a continuous–time controller of the form $C(s) = K(s+1)/(s+a)$ (without taking into account the effect of ZOH) to have $\zeta = 1/\sqrt{2}$ and $t_s = 0.5$.

   b) Obtain a discrete–time controller $D_1(z)$ by discretizing $C(s)$.

   c) Calculate the closed–loop pulse transfer function $H_1(z)$. Find the velocity error coefficient K_{v1} and the actual values of ζ and t_s.

   d) Now design a discrete–time controller $D_2(z)$ directly in z–domain to meet the design specifications. Calculate the corresponding closed–loop pulse transfer function $H_2(z)$ and the velocity error coefficient K_{v2}.

6. In the closed–loop system in Fig.1, $T = 0.04, G(s) = 1/s(s+1)$ and $F(s) = 1$. Design $D(z)$ to have $K_v = 10, \zeta = 0.5$. Calculate the pulse transfer function of the resulting closed–loop system and the actual value of ζ.

7. In the closed–loop system in Fig.1, $T = 0.1, G(s) = (s+10)/s(s+1)$ and $F(s) = 1$.

   a) Plot the root–locus for $D(z) = K$, and determine the maximum allowable gain for closed–loop stability.

   b) Design $D(z)$ to have a damping ratio of $\zeta = 0.5$. Obtain the closed–loop pulse transfer function.

E) State Space Analysis and Design

1. Given a discrete–time system $\mathcal{D} = (\Phi, \Gamma, C)$ with

$$\Phi = \begin{bmatrix} 0.50 & 0.70 & 0.40 & 0.10 \\ 0.50 & -0.10 & 0.60 & -0.90 \\ -0.50 & 0.40 & -0.30 & 0.90 \\ -0.25 & 0.95 & 0.50 & 0.90 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

   a) Check reachability and observability of $\mathcal{D}$.

   b) Characterize the subspaces that can be reached from the origin in $1, 2, 3$ and $n \geq 4$ steps.

   c) Obtain a minimal system $\bar{\mathcal{D}}$ which has the same transfer function as $\mathcal{D}$.

   d) Transform $\bar{\mathcal{D}}$ into controllable form.

   e) Transform $\bar{\mathcal{D}}$ into observable form.

   f) The output of $\bar{\mathcal{D}}$ to an input $u(k) = 1(k)$ is observed to have the values $y(0) = y(1) = 2$. Find $y(k), k \geq 0$.

2. Obtain minimal realizations of

   a) $H(z) = \frac{z^2+4z-3}{z^3-z^2+0.25z}$

   b) $H(z) = \frac{z^3+0.5z^2-0.5z+1}{z^3-0.5z^2-0.25z+0.125}$

F) Deadbeat Control

1. In the closed–loop system in Fig. 1, $G(s) = 1/(s+1), F(s) = e^{-Ts}$ and $r(t) = t1(t)$.

   a) Design the discrete controller $G_d(z)$ to have $y(kT) = kT,\ k \geq K$ for minimum K. (Note that due to nonunity feedback this does not mean that $e(kT) = 0,\ k \geq K$.)

   b) Calculate and plot $y(kT)$ and $e(kT)$.

2. In the closed–loop system in Fig. 1, $T = 0.1, G(s) = 1/(s+1), F(s) = 1$ and $D(z) = K(z-c)/(z-1)$.

   a) Find K and c so that the closed-loop system shows a deadbeat behavior (that is, $H(z)$ has all poles at $z = 0$).

   b) For $D(z)$ found in (a), find $H(z, \gamma)$.

3. In the closed–loop system in Fig. 1, $G(s) = 1/(s+1), F(s) = 1$ and a constant disturbance $d(t) = 1$ acts additively on the plant. Design $C(z)$ to have $\{y(kT)\}$ finite.

4. Given a continuous–time plant $G(s) = 1/(s+1)$.

   a) Design a discrete controller $D(z)$ such that the unity–feedback sampled–data system (with a sampling period T) shows a deadbeat response to a reference input $r(t) = 1(t)$. For the same controller, check if the response is ripple–free.

   b) Repeat (a) for $r(t) = t1(t)$.

   c) Use state–space techniques to design or to prove nonexistence of a ripple–free deadbeat controller for $r(t) = t1(t)$.

   d) Use the result of the paper "Ripple–free deadbeat control..." to design or to prove nonexistence of a ripple–free deadbeat state feedback controller for $r(t) = 1(t)$.

   e) Show that the state–feedback controller in (d) is equivalent to a discrete dynamic controller with input $e(k) = r(k) - y(k)$ and output $u(k)$. Find the (discrete) transfer function of this equivalent controller, and compare with the one in part (a).

   f) Repeat (d) for $r(t) = t1(t)$.

G) Generalized Hold Functions

1. In the following sample-and-generalized hold configuration $F(t)$ is a T-periodic function defined as

$$F(t) = \begin{cases} \alpha_F, & 0 \leq t < T/2 \\ \beta_F, & T/2 \leq t < T \end{cases}$$

Find the corresponding transfer function $F_e(s)$ of the equivalent configuration.

2. In the following sampled-data system $F(t)$ and $G(t)$ are piecewice constant generalized hold functions of the form given in Q.1, and $T = 1$. Find $\alpha_F, \beta_F, \alpha_G, \beta_G$ so that the closed-loop pulse transfer function is $H(z) = 1/z$. [Hint: You may use either the state-space formulation of the paper "Generalized Sampled-Data Hold Functions", or the result of Q.1 together with modified Z-transform.]

3. Given a continuous-time system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

a) Find a piecewise constant periodic generalized hold function of the form

$$f(t) = \begin{cases} \alpha, & 0 \leq t < 1/2 \\ \beta, & 1/2 \leq t < 1 \end{cases}$$

such that with the control $u(t) = f(t-k)y(k),\ k \leq t < k+1$, the discrete model of the system (corresponding to $T = 1$) has a deadbeat response to any initial state.

b) Repeat (a) by using a controller of the form

$$u(t) = h_1 y(k - 1/3) + h_2 y(k - 2/3) + h_3 y(k-1),\ k \leq t < k+1$$

4. A single-input/single-output continuous-time plant described by a triple (A_p, b, c_p^T) is required to show a deadbeat response to a reference input generated by an autonomous reference system described by $(A_r, 0, c_r^T)$. The controller consists of scalar generalized hold functions operating on the sampled values of the outputs of the plant and the reference system. Derive conditions for solvability of the problem, and give a procedure for the design of the generalized hold functions when these conditions are satisfied.

5. Given a continuous-time system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned}$$

   a) Design a deadbeat controller consisting of a generalized hold function operating on sampled outputs with $T_0 = 1$.
   b) Design a deadbeat controller consisting of a discrete dynamic system operating on multirate output feedback samples with frame period $T_0 = 1$ to generate an input sequence $\{u(kT_0)\}$.
   c) Repeat (b) if the discrete system in the controller itself is to have deadbeat dynamics.

6. A continuous–time plant with

$$A_p = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_p = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

is driven by a reference system with

$$A_r = 0\,, \quad C_r = 1$$

Check if ripple–free deadbeat control using constant state feedback from the sampled states of the plant and the reference system is possible. If your answe is positive, calculate the feedback gains for $T = 0.1$.

H) Multirate Sampled–Data Systems

1. Consider the plant in Q.G6.

   a) Using sampled–data output feedback and generalized hold functions with $T = 0.1$, place all the poles of the resulting closed–loop discrete system at $z = 0.5$.

   b) Repeat (a) by using a discrete controller with poles at $z = 0$ operating on multirate samples of the output $\{y((k-1)T), y((k-1)T+\tau), ..., y((k-1)T+(N-1)\tau)\}$ at time kT where $\tau = T/N$.

2. In the following closed–loop system $\dot{y}(t) = u(t)$. Obtain the pulse transfer function $Y_T(z)/R_T(z)$ by using

   a) the switch decomposition method

   b) the multirate impulse modulation method.

3. Using the notation of the paper "Multivariable, Multirate Sampled-Data Systems", write down the expressions $f^D(k), f^D(s), f^{DEV}(s), f^*(s)$ for the signal $f(t) = t1(t)$ and for $T = 1/3, T_o = 1 \;\; (N = 3)$.

4. Consider the following closed-loop multirate feedback system, where

   $$T_1 = 0.1,\; T_2 = 0.2,\; g(s) = 1/(s+1)$$

   a) Obtain an expression for $y_2^*(s)$ in terms of regular and modified impulse modulations of u_1.

   b) Calculate $Q_{21}^s(s)$ and $Q_{21}^{MIM}(s)$.

   c) Obtain an expression for $y_2^*(s)$ in terms of u_1, and compare the result with that of part (a). Hint:Use $y_{21}^{DEV}(s) = Q_{21}^{MIM}(s)u_1^{DEV}(s)$ and the relations between $y_2^*(s), u_1^*(s)$ and $y_2^{DEV}(s), u_1^{DEV}(s)$.

   d) Calculate $\Psi_{12}^s(s)$ and $\Psi_{12}^{MIM}(s)$.

   e) Obtain the closed-loop pulse transfer function relating $y_2^*(s)$ to $r_2^*(s)$ using the closed-loop relation between $y_2^{DEV}(s)$ and $r_2^{DEV}(s)$ in terms of $Q_{21}^{MIM}(s)$ and $\Psi_{12}^{MIM}(s)$, and the relations between $y_2^*(s), r_2^*(s)$ and $y_2^{DEV}(s), r_2^{DEV}(s)$.

   f) Repeat (e) using the switch decomposition method, and compare your result with that of part (e).

   g) Using Nyquist criterion determine the range of f for stability of the closed-loop system.