# Distributed Bounding of Feasible Sets in Cooperative Wireless Network Positioning

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***Abstract*—Locations of target nodes in cooperative wireless sensor networks can be confined to a number of feasible sets in certain situations, e.g., when the estimated distances between sensors are larger than the actual distances. Quantifying feasible sets is often challenging in cooperative positioning. In this letter, we propose an iterative technique to cooperatively outer approximate the feasible sets containing the locations of the target nodes. We first outer approximate a feasible set including a target node location by an ellipsoid. Then, we extend the ellipsoid with the measured distances between sensor nodes and obtain larger ellipsoids. The larger ellipsoids are used to determine the intersections containing other targets. Simulation results show that the proposed technique converges after a small number of iterations.**

***Index Terms*—Wireless sensor network, outer approximation, feasible sets, ellipsoid approximation, cooperative positioning.**

## I. Introduction

Wireless sensor networks (WSNs) rely on location information to tag sensed data with a geographical position [1]. In networks with a limited number of reference nodes, cooperative positioning can considerably enhance the quality of the location information [2]. Constraining the location of target nodes to some closed sets (feasible sets), can be incorporated into positioning algorithms, resulting in more accurate and robust estimates [3]. Quantifying such feasible sets is often a challenging task. The feasible region can also provide valuable informationl for evaluation of different services provided by WSNs and also for system design and resource management.

For noncooperative networks, a number of researchers propose techniques to outer approximate the feasible sets [4], [5]. For cooperative networks, [3], [6] employ a technique to cooperatively estimate locations of target nodes using outer approximation of feasible sets by discs (in 2D networks) through a heuristic approach. The method introduced in [6] has several drawbacks: first, the disc approximation of the intersection is not an efficient way to capture the structure of the intersection; second, the approach cannot easily be extended to 3D networks.

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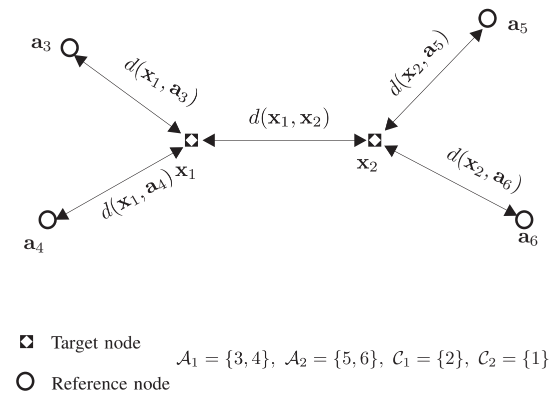


Fig. 1. A cooperative network with two targets and four reference nodes.

In this letter, in order to improve the outer approximation of the intersection considered in [6] and also to generalize the idea of outer approximation to 3D networks, we propose a technique based on convex optimization to cooperatively bound the feasible regions using ellipsoid approximations. Simulation results show that the proposed technique converges fast. Numerical results also confirm that the volumes of the resulting ellipsoids in cooperative scenarios are considerably smaller than the ones in noncooperative scenarios.

## II. System Model

We consider an $m$-dimensional network ($m = 2$ or 3) with $N + M$ nodes. Suppose that $M$ targets are placed at unknown positions $\mathbf{x}_i \in \mathbb{R}^m$, $i = 1, \ldots, M$, and $N$ reference nodes are located at known positions $\mathbf{a}_j \in \mathbb{R}^m$, $j = M+1, \ldots, M+N$. We define $\mathcal{A}_i = \{j|$ reference node $j$ can communicate with target $i\}$ and $\mathcal{C}_i = \{j|\ j \neq i$, target $j$ can communicate with target $i\}$ as the sets of indices of all reference and target nodes connected to target $i$ (see Fig. 1 for an example). For noncooperative networks, we set $\mathcal{C}_i = \emptyset$. The range estimate between sensor nodes is modeled as

$$\hat{d}_{ij} = d(\mathbf{x}_i, \mathbf{z}_j) + \epsilon_{ij},\ j \in \mathcal{A}_i \cup \mathcal{C}_i, \quad i = 1, \ldots, M, \qquad (1)$$

where $d(\mathbf{x}_i, \mathbf{z}_j) = \|\mathbf{x}_i - \mathbf{z}_j\|_2$ is the Euclidian distance between $\mathbf{x}_i$ and $\mathbf{z}_j$, $\epsilon_{ij}$ is the measurement error, and $\mathbf{z}_j = \mathbf{a}_j$ if $j \in \mathcal{A}_i$ or $\mathbf{z}_j = \mathbf{x}_j$ if $j \in \mathcal{C}_i$. Different distributions have been considered to model the measurement errors, e.g., Gaussian, uniform, exponential, or Laplacian [6]–[8]. In some scenarios the measured distances are larger than the actual distances, meaning that the measurement noise is nonnegative [6]. The nonnegative measurement assumption can be fulfilled in some cases, such as in non-line-of-sight conditions. In recent ultra-wide bandwidth measurements, it has been observed that the

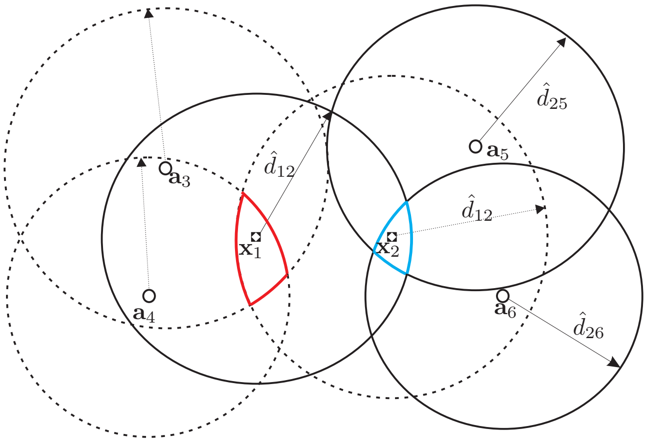


Fig. 2. For distance measurements with nonnegative errors, target nodes 1 and 2 for the network shown in Fig. 1 lie in closed bounded sets.

measurement noise tends to be positive [9]. In fact, time-of-arrival-based ranging typically involves setting a threshold such that false alarms (negative errors due to noise peaks) are negligible. Hence, negative ranging errors can be considered to occur very rarely, if at all. In this paper, we assume that measurement errors are nonnegative, i.e., $\epsilon_{ij} \geq 0$.

## III. Outer-approximation of feasible sets

### A. Implicit definition of feasible sets

Under the condition that the estimated distances are larger than the actual distances (i.e., $\epsilon_{ij} \geq 0$), we define the balls $\mathcal{B}_{ij}, i = 1, \ldots, M$, centered at $\mathbf{z}_j$ including the location of target node $i$ as follows:

$$\mathcal{B}_{ij} = \left\{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x} - \mathbf{z}_j\|_2 \leq \hat{d}_{ij}\right\}, \quad j \in \mathcal{A}_i \cup \mathcal{C}_i.$$

Hence, the location of target node $i$ belongs to

$$\mathbf{x}_i \in \mathcal{B}_i = \bigcap_{j \in \mathcal{A}_i \cup \mathcal{C}_i} \mathcal{B}_{ij}, \tag{2}$$

As an example, Fig. 2 shows how the feasible sets for target nodes 1 and 2 derived from distance measurements with nonnegative errors.We will assume that for every target $i$ at least one $\epsilon_{ij} \geq 0$, so that the feasible set has a nonempty interior. Our goal is to determine explicit expressions for $\mathcal{B}_i$ in (2). Since $\mathcal{B}_i$ can be a complex convex set, we resort to an ellipsoid outer approximation of $\mathcal{B}_i$, described in the next section.

*Remark 1:* It is observed that the volume of the intersection in (2) depends on the geometry of the network. For example, if a target lies outside the convex hull of its neighbors, the intersection containing the target location, and hence also the approximated intersection, can be large.

### B. Ellipsoid outer approximation

In Section III-C, we will propose a technique to outer approximate the feasible sets in a cooperative fashion. The idea is that for every target we find a convex set (an ellipsoid) guaranteed to contain the target location and then cooperatively shrink the ellipsoids. Before the detailed discussion in the next section, we first review two representations of an ellipsoid [10], [11]:

1) *A quadratic form:*

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{x}^T\mathbf{b} + c \leq 0\}, \tag{3}$$

where $\mathbf{A} \in \mathbb{S}_+^m$, where $\mathbb{S}_+^m$ is the set of $m$ by $m$ symmetric positive definite matrices, $\mathbf{b} \in \mathbb{R}^m$, and $c \in \mathbb{R}$. It is also required that $\mathbf{b}^T\mathbf{A}\mathbf{b} - c > 0$.

2) *An image of the unit ball*[1] *under an affine mapping:*

$$\mathcal{E} = \{\mathbf{P}\mathbf{x} + \mathbf{x}_c : \|\mathbf{x}\|_2 \leq 1,\ \mathbf{x} \in \mathbb{R}^m\}, \tag{4}$$

with $\mathbf{x}_c \in \mathbb{R}^m$ being the center of the ellipsoid and $\mathbf{P} \in \mathbb{S}_+^m$.

To derive (3) from (4), we can write

$$\mathbf{A} = \mathbf{P}^{-2}, \quad \mathbf{b} = -\mathbf{P}^{-2}\mathbf{x}_c, \quad c = \mathbf{x}_c^T\mathbf{P}^{-2}\mathbf{x}_c - 1. \tag{5}$$

The semi-axes of an ellipsoid are given by $\sqrt{\lambda_i}$, where $\lambda_i$ are the eigenvalues of the matrix $\mathbf{A}$ [10, Ch. 2]. To outer approximate the intersection by an ellipsoid, we first find the maximum volume ellipsoid contained in the intersection and then expand it to cover the intersection.[2]

### C. Proposed method

Consider the first representation in (3) for the $|\mathcal{A}_i \cup \mathcal{C}_i|$ ellipsoids $\mathcal{B}_{ij}$ in (2), the maximum volume ellipsoid contained in the intersection $\mathcal{B}_i$, expressed as (4), can be found by solving the following convex optimization problem [10]

$$\begin{aligned} &\underset{\mathbf{P}_i, \mathbf{x}_{c_i}, \boldsymbol{\tau}}{\text{maximize}}\ \log\det \mathbf{P}_i \\ &\text{subject to } \mathbf{U}_j \succeq 0,\ j = 1, \ldots, |\mathcal{A}_i \cup \mathcal{C}_i|,\ \boldsymbol{\tau} \geq \mathbf{0}, \end{aligned} \tag{6}$$

where $\mathbf{U}_j \succeq 0$ means that $\mathbf{U}_j$ is a positive semidefinite matrix, and is given by

$$\mathbf{U}_j = \begin{bmatrix} -\tau_j - c_j + \mathbf{b}_j^T\mathbf{A}_j^{-1}\mathbf{b}_j & \mathbf{0} & (\mathbf{x}_{c_i} + \mathbf{A}_j^{-1}\mathbf{b}_j)^T \\ \mathbf{0} & \tau_j\mathbf{I}_N & \mathbf{P}_i \\ \mathbf{x}_{c_i} + \mathbf{A}_j^{-1}\mathbf{b}_j & \mathbf{P}_i & \mathbf{A}_j^{-1} \end{bmatrix}.$$

The solution to the optimization problem in (6) gives the maximum volume ellipsoid (parametrized by $\mathbf{P}_i$ and $\mathbf{x}_{c_i}$) contained in the intersection of a number of ellipsoids. It was shown in [10, Ch. 8] that if we scale this ellipsoid around $\mathbf{x}_{c_i}$ by the dimension $m$, we obtain an ellipsoid that covers the intersection. Moreover, if the intersection is a symmetric set about a point, the scaling factor can be reduced to to $\sqrt{m}$ [10].

In practice, the sets $\mathcal{B}_{ij}$ for $j \in \mathcal{C}_i$ are not a priori available, since the positions of neighboring targets are unknown. Setting $\mathcal{B}_{ij} = \mathbb{R}^m$ for $j \in \mathcal{C}_i$ allows us to solve (6) based only on information from reference nodes (i.e., without cooperation), leading to an ellipsoid outer approximation parameterized $m\mathbf{P}_i^{(0)}, \mathbf{x}_{c_i}^{(0)}$. We can now iteratively improve the outer approximations as follows. Suppose that at the $k$-th iteration the ellipsoid outer approximation of the intersection (2) related to target $i$ is given by

$$\mathcal{E}_i^{(k)} = \left\{\bar{\mathbf{P}}_i^{(k)}\mathbf{x} + \mathbf{x}_{c_i}^{(k)} : \|\mathbf{x}\|_2 \leq 1\right\} \supseteq \mathcal{B}_i, \tag{7}$$

[1] A ball $\mathcal{B} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x} - \mathbf{a}\|_2 \leq R\}$ is a structured ellipsoid with $\mathbf{A} = \mathbf{I}$, $\mathbf{b} = -\mathbf{a}$, and $c = \|\mathbf{a}\|^2 - R$ in (3).

[2] The problem of finding the minimum volume ellipsoid covering the intersection of a number of ellipsoids is not tractable in general [11].

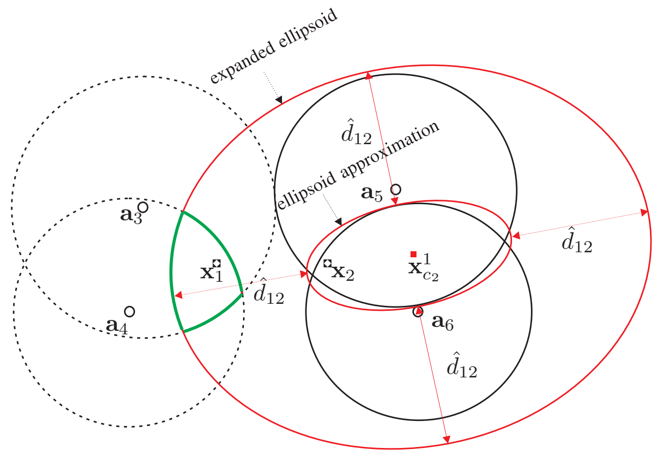


Fig. 3. Ellipsoid outer approximation of the intersection including target node 1 can be expanded to determine an intersection for target node 2.

where $\bar{\mathbf{P}}_i^{(k)} = m\mathbf{P}_i^{(k)}$. Now consider a target $\ell$ for which we want to improve the outer approximation. Suppose $i \in \mathcal{C}_\ell$, i.e., a neighbor node to the $\ell$-th node. We first expand the ellipsoid $\mathcal{E}_i^{(k)}$ uniformly in every direction with $\hat{d}_{\ell i}$. This is achieved by performing an eigenvalue decomposition of $\bar{\mathbf{P}}_i^{(k)}$ as

$$\bar{\mathbf{P}}_i^{(k)} = \mathbf{V}_i \mathbf{\Lambda}_i \mathbf{V}_i^T \tag{8}$$

and form

$$\mathbf{F}_{\ell i}^{(k)} = \mathbf{V}_i(\mathbf{\Lambda}_i + \hat{d}_{\ell i}\mathbf{I}_m)\mathbf{V}_i^T. \tag{9}$$

This leads to an expanded ellipsoid

$$\mathcal{F}_{\ell i}^{(k)} = \left\{ \mathbf{F}_{\ell i}^{(k)}\mathbf{x} + \mathbf{x}_{c_i}^{(k)} : \|\mathbf{x}\|_2 \le 1 \right\}, \tag{10}$$

which is guaranteed to contain $\mathbf{x}_\ell$ as well as $\mathbf{x}_i$. This procedure is applied to all neighbors of target $\ell$, so that

$$\mathbf{x}_\ell \in \mathcal{S}_\ell^{(k)} = \bigcap_{j\in\mathcal{A}_\ell} \mathcal{B}_{\ell j} \bigcap_{i\in\mathcal{C}_\ell} \mathcal{F}_{\ell i}^{(k)}. \tag{11}$$

Observe that $\mathcal{B}_{\ell j}$ is fixed, while $\mathcal{F}_{\ell i}^{(k)}$ is updated at every iteration. Node $\ell$ can find an outer approximation of $\mathcal{S}_\ell^{(k)}$ by solving a convex optimization problem of the form (6). Fig. 3 graphically shows how the ellipsoid outer approximation for target node 2 can be involved in determining the intersection for target node 1.

This procedure continues for a number of iterations to find ellipsoids covering the intersection for all target nodes.[3] Updating for the $\ell$-th target can be stopped after $K$ iterations if $\|[\bar{\mathbf{P}}_\ell^{(K)}\ \mathbf{x}_{c_\ell}^{(K)}] - [\bar{\mathbf{P}}_\ell^{(K-1)}\ \mathbf{x}_{c_\ell}^{(K-1)}]\|_{\mathrm{F}}$ is small enough, where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm.

The updating procedure can be performed in a sequential or parallel manner. In a sequential algorithm, once a target node $i$ determines an ellipsoid enclosing its intersection, it immediately broadcasts the parameters of the $i$-th ellipsoid, i.e., $\mathbf{P}_i^{(k)}$ and $\mathbf{x}_{c_i}^{(k)}$. Target nodes connected to node $i$ form new ellipsoids considering $\mathbf{P}_i^{(k)}$, $\mathbf{x}_{c_i}^{(k)}$, and $\hat{d}_{ji}$. Algorithm 1 implements the sequential algorithm. Note that Algorithm 1 can be considered as a geometric positioning algorithm. It can also provide geometric constraints to traditional positioning algorithms (e.g., least squares) to improve positioning accuracy.

[3]The convergence proof needs further exploration in future studies.

**Algorithm 1** *Cooperative outer-approximation*

Initialization: $\mathcal{F}_{ij}^{(0)} = \mathbb{R}^m,\ j \in \mathcal{C}_i,\ i = 1,\dots,M$
**for** $k = 0$ until convergence (or predefined $K$) **do**
  **for** $i = 1,\dots,M$ **do**
    determine ellipsoid outer approximation (EOA) of $\mathcal{S}_i^{(k+1)}$ using (6)

$$\left(\mathbf{x}_{c_i}^{(k+1)}, \mathbf{P}_i^{(k+1)}\right) := \mathrm{EOA}\Big\{ \bigcap_{j\in\mathcal{A}_i} \mathcal{B}_{ij} \bigcap_{j\in\mathcal{C}_i} \mathcal{F}_{ij}^{(k)} \Big\}$$

    form $\bar{\mathbf{P}}_i^{(k+1)} = m\mathbf{P}_i^{(k+1)}$ and decompose matrix $\bar{\mathbf{P}}_i^{(k+1)}$ (eigen decomposition) as $\bar{\mathbf{P}}_i^{(k+1)} = \mathbf{V}_i\mathbf{\Lambda}_i\mathbf{V}_i^T$
    **for** $\ell = 1,\dots,M$ **do**
      if $i \in \mathcal{C}_\ell$, then update the set $\mathcal{F}_{\ell i}^{(k+1)}$ as

$$\mathbf{F}_{\ell i}^{(k+1)} = \mathbf{V}_i(\mathbf{\Lambda}_i + \hat{d}_{\ell i}\mathbf{I}_m)\mathbf{V}_i^T,$$
$$\mathcal{F}_{\ell i}^{(k+1)} = \left\{ \mathbf{F}_{\ell i}^{(k+1)}\mathbf{x} + \mathbf{x}_{c_i}^{(k+1)} : \|\mathbf{x}\|_2 \le 1 \right\},$$

    **end for**
  **end for**
**end for**

## IV. Simulation Results

We consider the same network as in [6] with 13 reference nodes. For details of the network deployment please see [6]. A number of target nodes are randomly distributed inside the area. We assume two nodes are connected if the distance between them is equal to or smaller than $R_{\max}$. To evaluate the volume of an ellipsoid, parametrized with matrix $\mathbf{P}$, we consider $\det(\mathbf{P})$. Measurement noise is drawn from an exponential distribution [6] with a mean of 1 m. To solve the optimization problems formulated in this study, we use the CVX toolbox [12].

Fig. 4 illustrates an example in which the ellipsoid approximation of the intersections containing the target nodes (black stars), i.e., green ellipsoids, can be expanded to be involved in determining the intersection containing another target node (red triangle). We also plot the disc approximation from [6] of the intersection in both noncooperative (black dashed circle) and cooperative (black solid circle) modes. It is observed that the volume of the approximated ellipsoid in the cooperative mode (red solid ellipsoid) is considerably smaller than the one in the noncooperative scenario (red dashed ellipsoid). Moreover, the ellipsoid approximation approach results in a smaller volume than the disc approximation technique. In the simulations, the algorithm was run for 4 iterations, i.e., $K = 4$ in Algorithm 1.

Fig. 5 shows the average volumes of the ellipsoids covering the intersections versus the iteration number $k$ (outer loop iteration in Algorithm 1) for different numbers of target nodes. We observe that the algorithm converges quickly. It is also concluded that as more target nodes are involved, the outer approximation of the intersection gets smaller.

Finally to investigate the usefulness of the approximated intersection in positioning, we compare the performance of the semidefinite programming (SDP) relaxation technique [13] with a constrained least squares (CNLS) that combines the

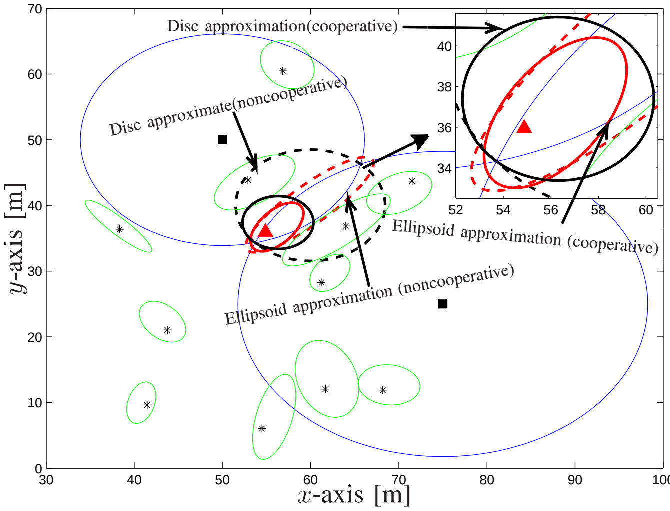


Fig. 4. The extended ellipsoid from ellipsoids containing targets (green ellipsoids, black stars) helps determining a smaller ellipsoid containing another target (red triangle) in cooperative scenario. Blue circles centered at reference nodes' locations (black squares) contain the red triangle target. The black dashed circle and black solid circle, respectively, show the disc approximation of the intersection in noncooperative and cooperative scenarios.

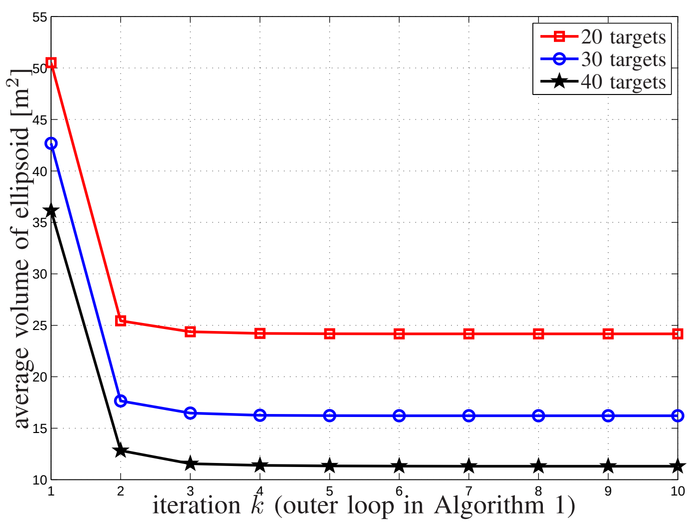


Fig. 5. The average volumes of approximated outer ellipsoids versus the number iterations for connectivity range $R_{\max} = 35$ m.

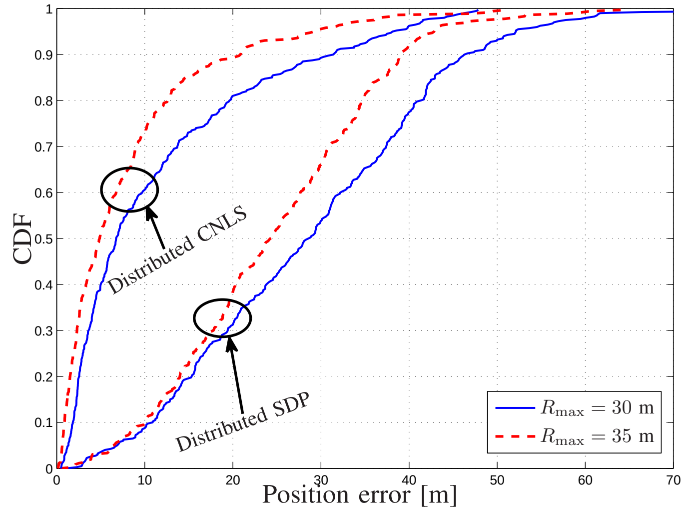


Fig. 6. The CDF of position error for CNLS and distributed SDP for 40 targets.

least squares algorithm with the constraints from the intersection involving the target nodes. Note that we implement both algorithms in a distributed fashion. That is, we first find an estimate of a target location (using SDP or CNLS) and consider the target node as a pseudo reference node in locating other targets. We update both algorithms for 10 iterations. Fig. 6 shows the cumulative distribution function (CDF) of position errors for CNLS and SDP for two different values of $R_{\max}$. As it is observed CNLS considerably outperforms the distributed SDP. Note that the original SDP, which has very good performance, is a centralized approach and an efficient version of distributed SDP may need further considerations.

## V. Conclusions

In this letter, we have considered cooperative positioning in wireless networks in which the estimated distances are larger than the actual distances. As a result, targets' locations can be confined to feasible (convex) sets. We have studied cooperative outer bounding of these feasible sets using ellipsoid outer approximations. The proposed approach can be implemented in a distributed manner. Simulation results show fast convergence of the proposed approach. One open problem for future studies is to prove to the convergence of the algorithm developed in this letter.

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