size decision regions centered at these levels and decision thresholds placed halfway between consecutive levels. In our case, the problem is complicated by the fact that the noise at the processor output is signal-dependent, calling for a more sophisticated quantization scheme.

Formally, for a given maximum tolerable average probability of error per signal level, denoted P_e, and for equal *a priori* probabilities $P(v_i) = 1/L$, $i = 1, 2, ..., L$, the maximum attainable accuracy can be found by solving for the maximum value of L in the equation

$$\frac{1}{L}\sum_{i=1}^{L-1}\left[\sum_{j=1}^{i}\int_{z_i}^{z_{i+1}} p_V(v\,|\,v_j)\,\mathrm{d}v + \sum_{j=i+1}^{L}\int_{z_{i-1}}^{z_i} p_V(v\,|\,v_j)\,\mathrm{d}v\right] = P_e, \quad p_V(z_i\,|\,v_i) = p_V(z_i\,|\,v_{i+1}) \tag{2}$$

where the choice of signal levels v_i, $i = 1, 2, ..., L$, and decision thresholds z_i, $i = 0, 1, ..., L$, subject to the constraints $V_{min} = z_0 \le v_1 \le z_1 \le v_2 \cdots \le v_{L-1} \le z_{L-1} \le v_L \le z_L = V_{max}$, comprise the optimal quantization scheme [10].

Alternatively, we can formulate the problem in the context of parameter estimation theory by ascribing an *a priori* parameter PDF $p(m_S)$ to the signal we wish to estimate. Upon the observation of k samples of the output signal, creating the sample vector $\underline{v}$ with a joint PDF $p(\underline{v}\,|\,m_S)$, we can form the *a posteriori* parameter PDF

$$p(m_S\,|\,\underline{v}) = \frac{p(\underline{v}\,|\,m_S)\,p(m_S)}{\int p(\underline{v}\,|\,m_S)\,p(m_S)\,\mathrm{d}m_S} \tag{3}$$

which we ideally expect and desire to satisfy $p(m_S\,|\,\underline{v}) \to \delta(m_S - \hat{m}_S)$ as $k \to \infty$. Here $\hat{m}_S$ is the true value of the signal [10]. In this approach, accuracy can be quantified by the Cramér–Rao lower bound on the variance of the estimate [11], which offers a tradeoff between accuracy and speed.

In both approaches, the difficulties due to the signal dependence of noise at the output can be alleviated by the use of suitable normalizing transforms which remove from the noise power the dependence on the signal mean [10, 12].

References

[1] Timuçin D A *et al.* 1994 *J. Opt. Soc. Am. A* **11** 560–571
[2] Timuçin D A *et al.* 1994 *J. Opt. Soc. Am. A* (to be submitted)
[3] McIntyre R J 1972 *IEEE Trans. Electron. Devices* **19** 703–713
[4] Saleh B E A 1978 *Photoelectron Statistics* (Berlin: Springer-Verlag)
[5] Teich M C *et al.* 1984 *J. Opt. Soc. Am. B* **1** 366–389
[6] Goodman J W 1985 *Statistical Optics* (New York: Wiley)
[7] Saleh B E A and Teich M C *Fundamentals of Photonics* (New York: Wiley)
[8] Papoulis A 1991 *Probability, Random Variables, and Stochastic Processes* 3rd ed. (New York: McGraw-Hill)
[9] Trushkin A V 1993 *IEEE Trans. Inform. Theory* **39** 1180–1194
[10] Timuçin D A *et al.* 1994 *J. Opt. Soc. Am. A* (to be submitted)
[11] Van Trees H L 1968 *Detection, Estimation, and Modulation Theory* Part I (New York: Wiley)
[12] Prucnal P R and Saleh B E A 1981 *Opt. Lett.* **6** 316–318

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Space-variant filtering in fractional Fourier domains

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Abstract

Signals with significant overlap in both the space and frequency domains may have little or no overlap in a fractional Fourier domain. Spatial filtering in these domains may allow us to eliminate distortion components which cannot be eliminated in the ordinary Fourier domain.

1 Introduction

Space-invariant filtering may be performed by multiplying the Fourier transform of the input signal by the Fourier transform of the impulse response. Recently we have discussed how various space-variant operations can be performed by multiplying with a filter function in a fractional Fourier domain [1]. These operations can be realized optically, because the fractional Fourier transform can be realized optically. One approach is based on the use of quadratic graded index media [2, 3, 4], whereas another is based on the use of bulk lenses [5]. The graded index approach is closely connected to the definition of the fractional Fourier transform in terms of its spectral decomposition, whereas the bulk implementation is closely connected to its definition in terms of its linear transform kernel [1].

The many mathematical properties of the fractional Fourier transform, its relation to the Wigner space-frequency distribution, wavelet transforms, and chirp basis expansions, its applications to signal processing, and issues relating to its optical implementation are discussed in the references. Due to limited space, we will here content ourselves with the presentation of two examples of how space-variant filtering can be achieved by applying simple binary masks in fractional Fourier domains. Among the many things we cannot mention, of particular interest is correlation in fractional Fourier domains and its application to pattern recognition.

2 Definition of the fractional Fourier transform

The ath order fractional Fourier transform of a function $f(\cdot)$ is denoted by $\mathcal{F}^a[f](x)$ and may be defined as [1, 6]:

$$(\mathcal{F}^a[f])(x) = \int_{-\infty}^{\infty} \frac{e^{i(\pi\hat{\phi}/4-\phi/2)}}{|\sin\phi|^{1/2}} \exp[i\pi(x^2\cot\phi - 2xx'\csc\phi + x'^2\cot\phi)]\, f(x')\,dx',$$

where $\phi = a\pi/2$ and $\hat{\phi} = \mathrm{sgn}(\sin\phi)$. Some of its properties are: i.) linearity; ii.) $\mathcal{F}^0$ and $\mathcal{F}^4$ correspond to the identity operation; iii.) $\mathcal{F}^1$ corresponds to the conventional Fourier transform; iv.) $\mathcal{F}^{a_1}\mathcal{F}^{a_2} = \mathcal{F}^{a_1+a_2}$.

One of the most important properties states that performing the ath fractional Fourier transform operation corresponds to rotating the Wigner distribution by an angle $\phi = a(\pi/2)$

in the clockwise direction. We are unable to discuss the Wigner distribution here, although it is necessary for a complete understanding of the filtering examples discussed below. The reader is encouraged to consult [1] and the references given there. Roughly speaking, the Wigner distribution of a function $f(\cdot)$, denoted by $W_f(x,\nu)$, can be interpreted as a function that indicates the distribution of the signal energy over space x and frequency ν. Defining the rotation operator $\mathbf{R}_\phi$ for two-dimensional functions, corresponding to a counterclockwise rotation by ϕ, the property mentioned above can be expressed as $W_{\mathcal{F}^a[f]}(x,\nu) = \mathbf{R}_{-\phi} W_f(x,\nu)$. Another version of this property [7] is $\mathcal{R}_\phi[W_f(x,\nu)] = |\mathcal{F}^a[f]|^2$, where the operator $\mathcal{R}_\phi$ is the Radon transform evaluated at the angle ϕ. The Radon transform of a two-dimensional function is its projection on an axis making angle ϕ with the x axis.

Applications other than that discussed in this paper may be found in the references.

3 The fractional Fourier transform in optics

3.1 Optical implementation of the fractional Fourier transform

Analog optical implementations of the fractional Fourier transform have already been presented. In [1, 2, 3, 4] we discussed the fractional Fourier transforming property of quadratic graded index media. Lohmann suggested two systems consisting of thin lenses separated by free-space [5]. That the two approaches were equivalent and represented the fractional Fourier transform as defined above was demonstrated in [8].

The fact that the fractional Fourier transform can be realized optically means that the filtering examples discussed below can also be realized optically. Experimental results may be found in [9].

3.2 The fractional Fourier transform as a tool for analyzing optical systems

In [10, 11] we show that there exists a fractional Fourier transform relation between the (appropriately scaled) optical amplitude distributions on two spherical reference surfaces with given radii and separation. It is possible to determine the order and scale parameters associated with this fractional transform given the radii and separation of the surfaces. Alternatively, given the desired order and scale parameters, it is possible to determine the necessary radii and separation.

This result provides an alternative statement of the law of propagation and allows us to pose the fractional Fourier transform as a tool for analyzing and describing a rather general class of optical systems.

One of the central results of diffraction theory is that the far-field diffraction pattern is the Fourier transform of the diffracting object. It is possible to generalize this result by showing that the field patterns at closer distances are the fractional Fourier transforms of the diffracting object [10].

More generally, in an optical system involving many lenses separated by arbitrary distances, it is possible to show that the amplitude distribution is continuously fractional Fourier transformed as it propagates through the system. The order $a(z)$ of the fractional transform observed at the distance z along the optical axis is a continuous monotonic increasing function. As light propagates, its distribution evolves through fractional transforms of increasing orders. Wherever the order of the transform $a(z)$ is equal to $4j+1$ for any integer j, we observe the Fourier transform of the input. Wherever the order is equal to $4j+2$, we observe an inverted image, etc. [10, 11]

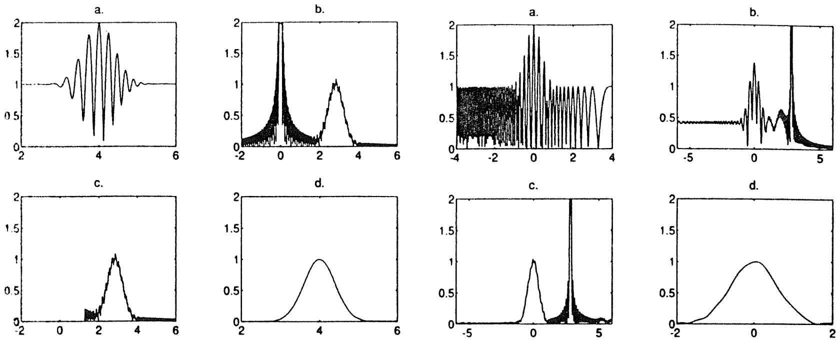

Figure 1: Example 1 (left) and example 2 (right)

4 Filtering examples

Consider the signal $\exp[-\pi(x-4)^2]$ distorted additively by $\exp(-i\pi x^2)\mathrm{rect}(x/16)$. The magnitude of their sum is displayed in part a., on the left hand side of the figure. These signals overlap in the frequency domain as well. In part b., we show their $a = 0.5$th fractional Fourier transform. We observe that the signals are separated in this domain. The chirp distortion is transformed into a peaked function which does not exhibit significant overlap with the signal transform, so that it can be blocked out by a simple mask (part c.). Inverse transforming to the original domain, we obtain the desired signal nearly perfectly cleansed of the chirp distortion (part d.).

Now we consider a slightly more involved example in which the distorting signal is also real. The signal $\exp(-\pi x^2)$ is distorted additively by $\cos[2\pi(x^2/2-4x)]\mathrm{rect}(x/8)$, as shown in part a., on the right hand side of the figure. The $a = 0.5$th transform is shown in part b. One of the complex exponential chirp components of the cosine chirp has been separated in this domain and can be masked away, but the other still distorts the transform of the Gaussian. After masking out the separated chirp component (not shown), we take the $a = -1$st transform (which is just an inverse Fourier transform) to arrive at the $a = -0.5$th domain (part c.). Here the other chirp component is separated and can be blocked out by another simple mask. Finally, we take the 0.5th transform to come back to our home domain (part d.), where we have recovered our Gaussian signal, with a small error.

The examples above have been limited to chirp distortions which are particularly easy to separate in a fractional Fourier domain (just as pure harmonic distortion is particularly easy to separate in the ordinary Fourier domain). However, it is possible to filter out more general types of distortion as well. In some cases this may require several consecutive filtering operations in several fractional domains of different order [1]. There is nothing special about our choice of Gaussian signals other than the fact that they allow easy analytical manipulation. Also, there is nothing special about the 0.5th domain. It just turns out that this is the domain of choice for the examples considered above.

In the above examples we have demonstrated that the method works, but did not discuss what led us to transform to a particular domain and what gave us the confidence that doing so will get rid of the distortion. This becomes very transparent once one understands the relationship between the fractional Fourier transform and the Wigner distribution. This relationship, as well as the general philosophy behind such filtering operations is discussed in [1].

5 Conclusion

What we know as the space and spatial frequency domains are merely special cases of fractional domains. These domains are indexed by the parameter a. The representation of a signal in the ath domain is the ath fractional Fourier transform of its representation in the $a = 0$th domain, which we define to be the space domain. The representation in the $a = 1$st domain is the conventional Fourier transform. If we set up a two-dimensional space, called the Wigner space, such that one axis (x) corresponds to the $a = 0$th domain (the conventional space domain) and the other (ν) to the $a = 1$st domain (the conventional spatial frequency domain), then the ath domain corresponds to an axis making an angle $\phi = a\pi/2$ with the x axis.

A desired signal and noise may overlap in both conventional space and frequency domains, but not in a particular fractional domain. Even when this is not the case, spatial filtering in a few fractional domains in cascade may enable the elimination of noise quite conveniently. It is possible to implement these operations optically, as well as with a fast digital algorithm.

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References

[1] H.M. Ozaktas, B. Barshan, D. Mendlovic and L. Onural, *J. Opt. Soc. Am.* A, 11:547-559, 1994

[2] H.M. Ozaktas and D. Mendlovic, *Opt. Commun.*, 101:163-169, 1993.

[3] D. Mendlovic and H.M. Ozaktas, *J. Opt. Soc. Am.* A, 10:1875-1881, 1993.

[4] H.M. Ozaktas and D. Mendlovic, *J. Opt. Soc. Am.* A, 10:2522-2531, 1993.

[5] A.W. Lohmann, *J. Opt. Soc. Am.* A, 10:2181-2186, 1993.

[6] A.C. McBride and F.H. Kerr, *IMA J. Appl. Math.*, 39:159-175, 1987.

[7] A.W. Lohmann and B.H. Soffer, *Technical Digest of the 1993 Annual Meeting of the OSA*, p.109, 1993.

[8] D. Mendlovic, H.M. Ozaktas and A.W. Lohmann, "Graded index fibres, Wigner distribution functions and the fractional Fourier transforms", to be published in *Appl. Opt.*.

[9] R.G. Dorsch, A.W. Lohmann, Y. Bitran, D. Mendlovic and H.M. Ozaktas, "Chirp filtering in the fractional Fourier domain: experimental results", to be published in *Appl. Opt.*.

[10] H.M. Ozaktas and D. Mendlovic, "The fractional Fourier transform as a tool for analysing beam propagation and spherical mirror resonators", to be published in *Opt. Lett.*.

[11] H.M. Ozaktas and D. Mendlovic, "Fractional Fourier Optics", submitted to *J. Opt. Soc. Am.* A.

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A novel form of incoherent optical correlator

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Abstract. Much work on digital signal processing, DSP, and optical signal processing, OSP, is aimed at pattern recognition. Here we describe a simple and rugged but promising hybrid digital/incoherent optical approach to this problem.

1. Introduction

The problem of identifying the presence of a given pattern in a scene is superficially similar to that of identifying a given radio or radar signal in a noisy environment. The optimum solution to the latter problem was devised in WW2 and takes the form of either a *correlator* or a *matched filter*, appropriately modified if the background noise does not have a white spectrum. For this reason the first and many subsequent approaches to pattern recognition have been based on these components.[1-4] However in reality there are numerous differences between these superficially similar problems. For example, the electronic problem is one-dimensional (time) while in pattern recognition we need to process a two-dimensional projection of a three-dimensional object, often with no a priori knowledge of its scale or orientation. In addition it turns out that the background spectrum is never white. Much of the research effort over the past thirty years has sought to overcome such complications of the pattern recognition problem. The work described here is a revisit of an earlier incoherent optical correlator, but with the incorporation of new features to improve its performance in various respects. In particular we describe a hybrid OSP/DSP processor incorporating spectral whitening which exploits each technology to advantage, and a multi-channel version of this device for use with scrolling input data/scenes.

2. Summary of new work on incoherent correlators

A simple and elegant incoherent correlator is described in various standard textbooks.[2,3] It operates by a shadow casting process based on geometrical optics, as shown schematically in Figure 1. As it stands this setup is non-ideal because all objects correlate with a given reference to some extent, the output deriving from optical intensity which is positive definite. This difficulty arises because the (spatial) spectrum of the scenery is not "white".