Fractional Fourier optics

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There exists a fractional Fourier-transform relation between the amplitude distributions of light on two spherical surfaces of given radii and separation. The propagation of light can be viewed as a process of continual fractional Fourier transformation. As light propagates, its amplitude distribution evolves through fractional transforms of increasing order. This result allows us to pose the fractional Fourier transform as a tool for analyzing and describing optical systems composed of an arbitrary sequence of thin lenses and sections of free space and to arrive at a general class of fractional Fourier-transforming systems with variable input and output scale factors.

Key words: Diffraction, Fourier optics, optical information processing, fractional Fourier transforms.

1. INTRODUCTION

The ath-order fractional Fourier transform $(\mathcal{F}^a \hat{q})(u)$ of the function $\hat{q}(u)$ is defined for $0 < |a| < 2$ as

$$(\mathcal{F}^a \hat{q})(u) \equiv \int_{-\infty}^{\infty} B_a(u, u') \hat{q}(u') \mathrm{d}u' ,$$

$$B_a(u, u') \equiv \frac{\exp[-i(\pi \hat{\phi}/4 - \phi/2)]}{|\sin \phi|^{1/2}}$$

$$\times \exp[i\pi(u^2 \cot \phi - 2uu' \csc \phi + u'^2 \cot \phi)], \quad (1)$$

where

$$\phi \equiv \frac{a\pi}{2} \qquad (2)$$

and $\hat{\phi} = \operatorname{sgn}(\sin \phi)$. The kernel is defined separately for $a = 0$ and $a = \pm 2$ as $B_0(u, u') \equiv \delta(u - u')$ and $B_{\pm 2}(u, u') \equiv \delta(u + u')$, respectively.[1,2] One may easily extend the definition outside the interval $[-2, 2]$ by noting that $\mathcal{F}^{4j+a} \hat{q} = \mathcal{F}^a \hat{q}$ for any integer j. Both u and u' are interpreted as dimensionless variables.

Some essential properties of the fractional Fourier transform are as follows: (i) It is linear. (ii) The first-order transform ($a = 1$) corresponds to the common Fourier transform. (iii) It is additive in index, $\mathcal{F}^{a_1} \mathcal{F}^{a_2} \hat{q} = \mathcal{F}^{a_1 + a_2} \hat{q}$. Other properties may be found in Refs. 1–11.

Optical implementations of the fractional Fourier transform have already been presented. In Refs. 3–7 we discussed the fractional Fourier-transforming property of quadratic graded-index media. Lohmann suggested two systems consisting of thin lenses separated by free space.[9] The fact that the two approaches were equivalent and represented the fractional Fourier transform as defined in Refs. 1 and 2 was demonstrated in Ref. 10. Applications have been suggested in these references and in Refs. 7, 11, and 12. Later research[13,14] provided certain extensions and experimental verification of the above results.

In this paper we derive a fundamental result stating that there exists a fractional Fourier-transform relation between the (appropriately scaled) optical amplitude distributions on two spherical reference surfaces with given radii and separation (Fig. 1 below). This result provides an alternative statement of the law of propagation and allows us to pose the fractional Fourier transform as a tool for analyzing and describing a rather general class of optical system. (Previous research has been primarily concerned with offering optical realization of the fractional Fourier transform. These realizations follow as special cases or applications of our more general formulation, which allows us to state the necessary and sufficient conditions for a fractional Fourier transform in full generality.)

After discussing in some detail the above-mentioned result, we show how the Fresnel diffraction integral can be expressed in terms of a fractional Fourier transform. We discuss how axially centered systems composed of an arbitrary number of lenses separated by arbitrary distances can be analyzed by means of fractional Fourier transforms. As an instructive example, we concentrate on the classical single-lens imaging configuration. We also specify the general conditions under which an arbitrary system is a fractional Fourier transformer.

Whenever we can express the result of an optical problem (such as Fraunhofer diffraction) in terms of a Fourier transform, we tend to think of this result as simple and elegant. This is justified by the fact that the Fourier transform has many simple and useful properties that make working with it attractive. The Fourier transform and image occur at certain privileged planes in an optical system. Often all our intuition about what happens in between these planes is that the amplitude distribution is given by a complicated integral. In this paper we show that the distribution of light at intermediate planes can be expressed in terms of the fractional Fourier transform (which also has several useful properties and operational formulas). Thus the fractional Fourier transform

completes in a natural way the study of optical systems often called Fourier optics.

We note in passing that, although one-dimensional signals are considered throughout this paper for notational simplicity, straightforward generalization of the results to two dimensions is possible. Although for one-dimensional systems it is more correct to speak of circular (or cylindrical) surfaces, here we customarily speak of spherical surfaces.

As a final word on terminology, we believe that ultimately the term Fourier transform should mean, in general, fractional Fourier transform and that the currently standard Fourier transform should be referred to as the first-order Fourier transform. Likewise, DFT should denote the discrete (fractional) Fourier transform, etc., and the invention of new acronyms and abbreviations should be discouraged. Then, instead of speaking of fractional Fourier optics, we will be able to speak simply of Fourier optics.

2. FRACTIONAL FOURIER-TRANSFORMING PROPERTY OF PROPAGATION THROUGH FREE SPACE

We refer to Fig. 1. The complex amplitude distributions with respect to the first and the second spherical reference surfaces are denoted by $q_1(x')$ and $q_2(x)$, respectively. The distributions with respect to the planar surfaces tangent to the spherical surfaces on the optical axis are likewise denoted by $p_1(x')$ and $p_2(x)$. If the radii of the spherical surfaces are denoted by R_1 and R_2, we have

$$p_2(x) = q_2(x)\exp(i\pi x^2/\lambda R_2), \tag{3}$$

$$p_1(x') = q_1(x')\exp(i\pi x'^2/\lambda R_1), \tag{4}$$

where λ is the wavelength. Assuming propagation from left to right, $p_2(x)$ is related to $p_1(x')$ by a Fresnel integral:

$$p_2(x) = \frac{\exp(i2\pi d/\lambda)}{\sqrt{i\lambda d}} \int_{-\infty}^{\infty} \exp[i\pi(x-x')^2/\lambda d]p_1(x')\mathrm{d}x'. \tag{5}$$

Combining Eqs. (3)–(5), we can obtain a relation between $q_1(x')$ and $q_2(x)$. To permit comparison of this relation with Eq. (1) we introduce the dimensionless variables $u' \equiv x'/s_1$ and $u \equiv x/s_2$, where s_1 and s_2 are real-valued scale parameters with dimensions of length. Also introducing the hatted functions $\hat{q}_1(u') \equiv q_1(u's_1)$ and $\hat{q}_2(u) \equiv q_2(us_2)$, we obtain

$$\hat{q}_2(u) = \frac{\exp(i2\pi d/\lambda)s_1}{\sqrt{i\lambda d}}$$
$$\times \int_{-\infty}^{\infty} \exp\left[\frac{i\pi}{\lambda d}(g_2 s_2^2 u^2 - 2s_1 s_2 uu' + g_1 s_1^2 u'^2)\right]$$
$$\times \hat{q}_1(u')\mathrm{d}u', \tag{6}$$

where we established the definitions

$$g_1 \equiv 1 + d/R_1, \tag{7}$$

$$g_2 \equiv 1 - d/R_2. \tag{8}$$

Now, comparing this result with the definition of the fractional Fourier transform [Eq. (1)], we conclude that

$\hat{q}_2(u)$ is proportional to the fractional Fourier transform of $\hat{q}_1(u')$, i.e.,

$$\hat{q}_2(u) = \left\{\frac{\exp[i2\pi d/\lambda]s_1 \exp[i(\pi\hat{\phi}/4 - \phi/2)]|\sin \phi|^{1/2}}{\sqrt{i\lambda d}}\right\}$$
$$\times (\mathcal{F}^a \hat{q}_1)(u), \tag{9}$$

if and only if

$$g_2 \frac{s_2^2}{\lambda d} = \cot \phi, \tag{10}$$

$$g_1 \frac{s_1^2}{\lambda d} = \cot \phi, \tag{11}$$

$$\frac{s_1 s_2}{\lambda d} = \csc \phi. \tag{12}$$

These three equations are the necessary and sufficient conditions for Eq. (9) to hold.

We now discuss the consequences of these equations from three perspectives.

A. Analysis

1. Problem
Given R_1, R_2, and d, find s_1, s_2, and a (or, equivalently, $\phi \equiv a\pi/2$). That is, we are given the reference surfaces and wish to find the order and the scale parameters of the resulting transform.

Equations (10) and (11) imply that

$$g_1 s_1^2 = g_2 s_2^2. \tag{13}$$

Now, using the identity $\cot^2 \phi + 1 = \csc^2 \phi$ and Eqs. (10)–(13), we obtain

$$s_2^4 = (\lambda d)^2(g_2/g_1 - g_2^2)^{-1}, \tag{14}$$

$$s_1^4 = (\lambda d)^2(g_1/g_2 - g_1^2)^{-1}. \tag{15}$$

Note that Eq. (13) implies that $g_1 g_2 \geq 0$ and that Eqs. (14) and (15) imply that $(g_2/g_1 - g_2^2) \geq 0$ and $(g_1/g_2 - g_1^2) \geq 0$. These conditions can be summarized in the form

$$0 \leq g_1 g_2 \leq 1. \tag{16}$$

If a fractional transform relation is to hold, R_1, R_2, and d must be specified such that this condition holds. (It
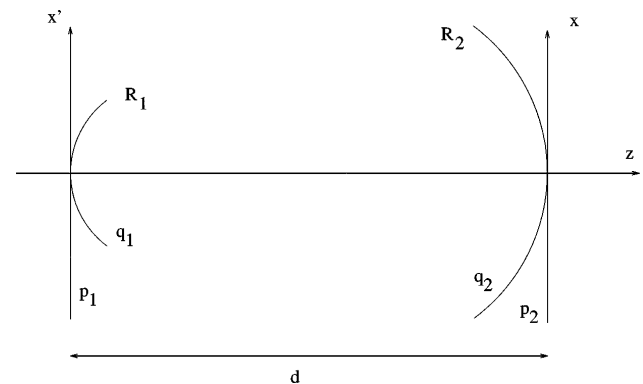

Fig. 1. Two spherical surfaces. This figure is drawn such that $R_1 < 0$ and $R_2 > 0$. The distance d is always taken to be positive.

is interesting that if Fig. 1 is interpreted as a spherical mirror resonator, this equation is the stability or the confinement condition of the resonator.[15])

We can see that, given R_1, R_2, and d, $|s_2|$ and $|s_1|$ are uniquely determined by Eqs. (14) and (15). Then Eq. (10) or (11) enables us to determine ϕ according to

$$\tan \phi = \pm[(1/g_1 g_2) - 1]^{1/2}, \qquad (17)$$

where the $\pm$ is determined according to the common sign of g_1 and g_2. The ambiguity in the inverse tangent function is resolved by examination of the sign of csc ϕ. Equation (12) tells us that, if we choose the signs of s_1 and s_2 such that $s_1 s_2 \geq 0$, then csc $\phi \geq 0$, so that ϕ lies in $[0, \pi]$. In contrast, if $s_1 s_2 \leq 0$, then csc $\phi \leq 0$, so that ϕ lies in $[-\pi, 0]$. These results are consistent with inversion and parity properties of the fractional Fourier transform.

2. Result
A fractional Fourier-transform relation exists between two spherical surfaces of radii R_1, R_2 and separation d if and only if $0 \leq g_1 g_2 \leq 1$. If this condition is satisfied, $|s_1|$ and $|s_2|$ are determined by Eqs. (14) and (15), and ϕ is determined within $\pm\pi$ by Eq. (17). The quadrant of ϕ is determined by our choice of the signs of s_1 and s_2, or vice versa.

B. Synthesis

1. Problem
Given s_1, s_2, and a (or, equivalently, $\phi \equiv a\pi/2$), find R_1, R_2, and d. That is, we wish to design a fractional Fourier-transform system with specific order and scale factors.

Notice that Eq. (12) implies that the sign of $s_1 s_2$ must be the same as the sign of csc ϕ, which is the same as the sign of ϕ. If s_1, s_2, and ϕ have been specified consistent with this requirement, Eq. (12) determines d:

$$d = (s_1 s_2/\lambda)\sin \phi . \qquad (18)$$

Then Eqs. (10) and (11) give g_1 and g_2, which in turn give R_1 and R_2:

$$1 + d/R_1 \equiv g_1 = (s_2/s_1)\cos \phi , \qquad (19)$$

$$1 - d/R_2 \equiv g_2 = (s_1/s_2)\cos \phi . \qquad (20)$$

Note that g_1 and g_2, as given by these equations, will always satisfy $g_1 g_2 = \cos^2 \phi$, and thus $0 \leq g_1 g_2 \leq 1$.

2. Result
A fractional Fourier-transform relation of order a between two spherical surfaces with the input and the output scaled by s_1 and s_2, respectively, can be obtained if and only if $\text{sgn}(s_1 s_2) = \text{sgn}(\sin \phi)$. If this condition is satisfied, we must choose R_1, R_2, and d according to Eqs. (18)–(20).

C. Propagation

1. Problem
Given s_1, R_1, and d, find a (or ϕ), s_2, and R_2. That is, given the radius of the spherical reference surface and

the scale parameter on the input side, find them at a distance d to the right, as well as the order of the resulting transform at that distance.

Because R_1 and d are given, g_1 is also known. Using Eqs. (10)–(12), we can obtain

$$\tan \phi = \frac{\lambda d}{g_1 s_1{}^2}, \qquad (21)$$

$$s_2{}^2 = g_1{}^2 s_1{}^2 + \frac{(\lambda d)^2}{s_1{}^2}, \qquad (22)$$

$$1 - d/R_2 \equiv g_2 = \frac{g_1 s_1{}^4}{g_1{}^2 s_1{}^4 + (\lambda d)^2} . \qquad (23)$$

We are free to choose the sign of s_2, which together with the specified sign of s_1 determines the sign of csc ϕ. This sign determines the quadrant of ϕ in Eq. (21). If we assume that s_1, s_2 are both positive, ϕ lies in the interval $[0, \pi]$. Now ϕ, as given by Eq. (21), is a continuous monotonic increasing function of d. Let us assume that the location of the first surface is fixed at the origin $z = 0$ of the optical axis and that we examine the distribution of light after it has propagated a distance $z = d$ in the positive z direction. For larger values of d, we observe fractional transforms of the initial distribution of larger-order ϕ; the amplitude distribution of light is continuously fractional Fourier transformed as it propagates.

2. Result
Given the distribution of light on a spherical reference surface of radius R_1 and scale parameter s_1, we can observe its fractional Fourier transform on another spherical reference surface a distance d to the right. The radius R_2 and scale parameter s_2 for this surface and the order of the fractional transform are given by Eqs. (21)–(23), where the quadrant of ϕ is determined by our choice of the sign of s_2.

3. ILLUSTRATIVE APPLICATIONS

We now consider some applications of the above results. For convenience we restrict s_1 and s_2 to positive values. This implies csc $\phi > 0$, so that ϕ lies in the interval $[0, \pi]$. Using Eq. (12) we can now write Eq. (9) in the more meaningful form

$$\hat{q}_2(u) = [\exp(i2\pi d/\lambda)\exp(-ia\pi/4)\sqrt{s_1/s_2}] \quad (\mathcal{F}^a \hat{q}_1)(u). \qquad (24)$$

The phase factor $\exp(i2\pi d/\lambda)$ is associated with propagation over the distance d. The phase factor $\exp(-i\phi/2)$ is the Gouy phase shift.[15] The factor $\sqrt{s_1/s_2}$ ensures power conservation.

A. Fresnel Diffraction as a Fractional Fourier Transform
Let us assume that a plane wave of unit amplitude illuminates a planar screen with complex amplitude transmittance $t(x)$. We can handle this case by letting $R_1 \to \infty$, as given in Fig. 1. We then have $q_1(x') = p_1(x') = t(x')$. Because $g_1 = 1$ in this case, Eq. (11) implies that cot $\phi \geq 0$, so that ϕ lies in the interval $[0, \pi/2]$.

We also assume that the scale parameter s_1 associated with the planar reference plane is specified freely. Then Eq. (11) gives us the order of the fractional transform observed at a distance d from the screen. The scale s_2 of the transform at this distance can then be found from Eq. (12). Then Eq. (10) yields g_2 and hence R_2. Thus the observed field $p_2(x)$ at a distance $d > 0$ is given by

$$p_2(x) = [\exp(i2\pi d/\lambda)\exp(-ia\pi/4)\sqrt{s_1/s_2}]\exp[(i\pi x^2/\lambda R_2)]$$
$$\times (\mathcal{F}^a \hat{t})(x/s_2), \qquad (25)$$

with

$$\frac{a\pi}{2} \equiv \phi = \arctan\left(\frac{\lambda d}{s_1^2}\right), \qquad (26)$$

$$s_2 = s_1\left[1 + \frac{(\lambda d)^2}{s_1^4}\right]^{1/2}, \qquad (27)$$

$$R_2 = d\left[1 + \frac{s_1^4}{(\lambda d)^2}\right], \qquad (28)$$

where $\hat{t}(u') \equiv t(u's_1)$. We can eliminate the spherical phase factor $\exp(i\pi x^2/\lambda R_2)$ by choosing a spherical reference surface with radius R_2, i.e., if we observe $g_2(x)$ instead of $p_2(x)$. (The various phase factors would have no effect if we were observing only the intensity.)

Thus we conclude that the Fresnel diffraction integral can be formulated as a fractional Fourier transform. As d is increased from 0 to ∞, the order a of the fractional transform increases according to Eq. (26) from 0 to 1. Letting $d \to \infty$, we obtain the intuitively appealing $a = 1$, $s_2 = \lambda d/s_1$, and $R_2 = d$, which we readily associate with the Fraunhofer diffraction pattern, which is nothing but the Fourier transform of the diffracting screen.

B. Symmetric Case

Referring to Fig. 1, let us consider the special case in which $-R_1 = R_2 \equiv R$ and $g_1 = g_2 \equiv g \equiv 1 - d/R$. Equations (10) and (11) then imply that $s_1 = s_2$, which we denote by s. The condition $0 \le g_1 g_2 \le 1$ becomes $0 \le g^2 \le 1$ or, more simply, $|g| \le 1$. This result implies that $0 \le d/R \le 2$ and hence that $R \ge d/2$.

Equations (10) and (12) now become

$$g(s^2/\lambda d) = \cot \phi, \qquad (29)$$

$$s^2/\lambda d = \csc \phi, \qquad (30)$$

from which it also follows that

$$g = \cos \phi. \qquad (31)$$

Given R and d such that $|g| \le 1$, Eq. (31) immediately determines ϕ. Then either Eq. (29) or (30) yields s. Alternatively, given ϕ and s, we can use the same equations to find d and g and hence R.

C. Fractional Fourier Transform between Planar Surfaces

We have seen that there exists a fractional Fourier-transform relation between two spherical surfaces, as depicted in Fig. 1. By using a lens to compensate the spherical phase factors at both surfaces, we can obtain a fractional Fourier transform between two planar surfaces.

We simply choose lenses with focal lengths $f_1 = -R_1$ and $f_2 = R_2$. Thus, with our synthesis result, it is possible to design a fractional Fourier transformer of given order a and of desired input and output scale parameters s_1 and s_2.

Let us restrict ourselves to the symmetric case for which $R \ge d/2 > 0$. Then, using two positive lenses of focal length $f = R > 0$, we obtain Lohmann's Type II fractional Fourier-transforming system (Fig. 2).[9] If the order of the fractional transform $a\pi/2 \equiv \phi$ and the scale s are specified, then the separation of the lenses d and their focal length f must be chosen as

$$d = (s^2/\lambda)\sin \phi, \qquad (32)$$

$$f = (s^2/\lambda)\cot(\phi/2). \qquad (33)$$

Alternatively, let us consider an asymmetric pair of spherical surfaces with $R_1 \to \infty$ and $R_2 = R$. Let us place immediately to the right of the second surface a thin lens of focal length $f = R/2$, whose effect will be to map the amplitude distribution on the spherical surface of radius R onto a spherical surface of radius $-R$. Now let us place after the lens a second asymmetric pair of spherical surfaces with $R_1 = -R$ and $R_2 \to \infty$. The overall system consists of a stretch of free space followed by a lens followed by another stretch of free space, which is nothing but Lohmann's Type I fractional Fourier-transforming system (Fig. 3). Its analysis is similar to what has been presented above and is thus not repeated.

We refer to such realizations of the fractional Fourier transform as canonical realizations Type II and Type I.
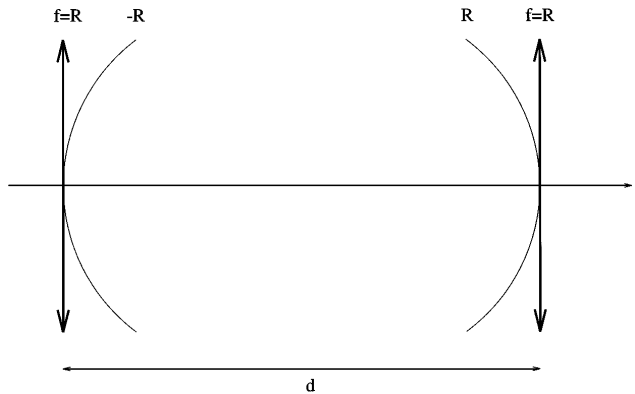

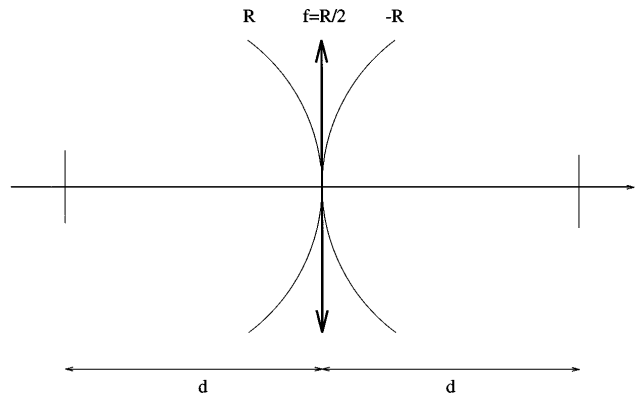Fig. 2.   Canonical realization Type II.

Fig. 3.   Canonical realization Type I.

They are summarized below within the framework of this paper. An alternative discussion of these systems is to be found in Lohmann's paper.[9]

Canonical fractional Fourier-transforming configurations have planar input and output reference surfaces separated by a distance l and with scale parameter s. The focal lengths of the lenses are denoted by f. The output p_{out} of such a system is related to the input p_{in} by the relation

$$\hat{p}_{\text{out}}(u) = \exp(i2\pi l/\lambda)\exp(-ia\pi/4)(\mathcal{F}^a \hat{p}_{\text{in}})(u), \quad (34)$$

where $u \equiv x/s_2$, $u' \equiv x'/s_1$ and $\hat{p}_{\text{out}}(u) \equiv p_{\text{out}}(us_2)$, $\hat{p}_{\text{in}}(u') \equiv p_{\text{in}}(u's_1)$. We can see that the output is essentially the ath-order fractional Fourier transform of the input.

1. Type II
A Type II system consists of a lens of focal length f followed by a section of free space of length d followed by a second lens of focal length f (Fig. 2). Thus its total length is $l = d$. We can obtain d and f in terms of s and ϕ (or a) by using

$$d = (s^2/\lambda)\sin \phi, \quad (35)$$

$$f = (s^2/\lambda)\cot(\phi/2). \quad (36)$$

Alternatively, provided that $f \geq d/2$, we can find ϕ (or a) and s in terms of d and f by using

$$\phi = \arccos(1 - d/f), \quad (37)$$

$$s^4 = \frac{\lambda^2 df}{2 - d/f}. \quad (38)$$

2. Type I
A Type I system consists of a section of free space of length d followed by a lens of focal length f followed by a second section of free space of length d (Fig. 3). Thus its total length is $l = 2d$. We can obtain d and f in terms of s and ϕ (or a) by using

$$d = (s^2/\lambda)\tan(\phi/2), \quad (39)$$

$$f = (s^2/\lambda)\csc \phi. \quad (40)$$

Alternatively, we can find ϕ (or a) and s in terms of d and f by using

$$\phi = \arccos(1 - d/f), \quad (41)$$

$$s^4 = \lambda^2 df(2 - d/f). \quad (42)$$

D. Classical Single-Lens Imaging
Consider the classical single-lens imaging configuration in which the object is located a distance $d_0 > 0$ to the left of the lens and the image is located a distance $d_i > 0$ to the right of the lens, which has focal length $f > 0$ (Fig. 4). We can view this system as performing two consecutive fractional Fourier-transform operations (from the object to the lens and from the lens to the image), provided that the radius R_- of the spherical reference surface just before the lens and the radius R_+ of the spherical reference surface just after the lens are related by

$$1/R_+ = (1/R_-) - (1/f). \quad (43)$$

This equation states that the amplitude distribution of light on the reference surface with radius R_- is mapped by the lens onto a reference surface with radius R_+. If we are to look into the combined effect of the two consecutive fractional transforms, we should choose the scale factors s_- and s_+ immediately before and immediately after the lens such that

$$s_+ = s_-. \quad (44)$$

Letting $\phi_o \equiv a_o\pi/2$ and $\phi_i \equiv a_i\pi/2$ denote the order of transformation occurring from the object to the lens and from the lens to the image, respectively, we can write the imaging condition (for an inverted image) as $a_o + a_i = \pm 2$ or, equivalently,

$$\phi_o + \phi_i = \pm \pi. \quad (45)$$

This condition follows from the fact that $\mathcal{F}^{\pm 2}[\hat{q}(u)] = \hat{q}(-u)$ for any $\hat{q}(u)$.

Now, with the definitions $g_- \equiv 1 - d_o/R_-$, $g_+ \equiv 1 + d_i/R_+$, we have [see Eqs. (10) and (11)]

$$\cot \phi_o = g_-(s_-{}^2/\lambda d_o), \quad (46)$$

$$\cot \phi_i = g_+(s_+{}^2/\lambda d_i). \quad (47)$$

It is now possible to show that Eq. (45) implies the well-known imaging condition

$$1/f = (1/d_o) + (1/d_i). \quad (48)$$

Furthermore, if we let s_o and s_i denote the scale factors associated with the object and the image, Eq. (12) lets us write

$$s_o s_-/\lambda d_o = \csc \phi_o, \quad (49)$$

$$s_+ s_i/\lambda d_i = \csc \phi_i. \quad (50)$$

Now, because Eq. (45) implies that $\csc \phi_o = \csc \phi_i$, we obtain the magnification M of the scale factors

$$M \equiv s_i/s_o = d_i/d_o, \quad (51)$$
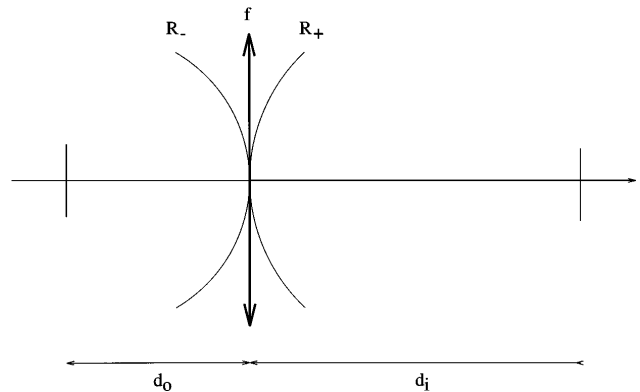
again a familiar result.

Fig. 4.   Single-lens imaging.

E.   General Lens Systems Analyzed as Consecutive Fractional Fourier Transforms

More-complicated systems involving several lenses separated by arbitrary distances can be analyzed in a similar manner. For simplicity let us restrict ourselves to positive lenses. Let the distance separating the input (or the object) from the first lens be d_0; the distance separating the first and the second lens, d_1; and the distance separating the ith and the $(i + 1)$th lens, d_i. The focal length of the ith lens is denoted by f_i. In general, the subscripts $i-$ and $i+$ are used to denote quantities immediately to the left and the right of the ith lens, respectively. ϕ_i denotes the order of the fractional transform associated with propagation from lens i to lens $i + 1$.

Assume that the input is specified with respect to a particular spherical reference surface of radius R_{0+} and that the input scale parameter is denoted by s_{0+}. Just before the first lens in the system we observe the fractional Fourier transform of the input of order ϕ_0, where ϕ_0 can be calculated from $\cot \phi_0 = g_{0+}s_{0+}{}^2/\lambda d_0$ [Eq. (11)], with $g_{0+} \equiv 1 + d_0/R_{0+}$. The scale of this transform is $s_{1-} = (\lambda d_0/s_{0+})\csc \phi_0$ [Eq. (12)], and the transform is observed on a spherical reference surface of radius R_{1-}, which we can calculate from $g_{1-} \equiv 1 - d_0/R_{1-}$ and $g_{1-} = (\lambda d_0/s_{1-}{}^2)\cot \phi_0$ [Eq. (10)].

Now we can make our way through the lens, using $s_{1+} = s_{1-}$ and $1/R_{1+} = 1/R_{1-} - 1/f_1$ [Eqs. (43) and (44)]. Then, just before the second lens, on a spherical reference surface of radius R_{2-}, we observe the fractional Fourier transform of the input of order $\phi_0 + \phi_1$, and so on.

Let us assume that all $d_0, d_1, \ldots, f_1, f_2, \ldots$, as well as s_{0+} and R_{0+}, are specified. Then we can work our way through the system iteratively, using the following equations for $i = 0, 1, 2, \ldots$:

$$g_{i+} = 1 + d_i/R_{i+}, \tag{52}$$

$$\cot \phi_i = g_{i+}s_{i+}{}^2/\lambda d_i, \tag{53}$$

$$s_{(i+1)-} = (\lambda d_i/s_{i+})\csc \phi_i, \tag{54}$$

$$s_{(i+1)+} = s_{(i+1)-}, \tag{55}$$

$$g_{(i+1)-} = [\lambda d_i/s_{(i+1)-}{}^2]\cot \phi_i, \tag{56}$$

$$R_{(i+1)-} = \frac{d_i}{1 - g_{(i+1)-}}, \tag{57}$$

$$R_{(i+1)+} = \frac{f_{i+1}R_{(i+1)-}}{f_{i+1} - R_{(i+1)-}}. \tag{58}$$

Here s_{i+} and R_{i+} may be considered to be the state variables of the iteration. The cumulative order of the transform just before the ith lens is $\phi_{\text{cum}_i} = \phi_0 + \phi_1 + \cdots + \phi_{i-1}$. This procedure allows us to find the orders of the transforms at the lenses, but of course it is also possible to calculate the order of the transform observed at any intermediate location (the propagation result).

Thus we can see that, in an optical system involving many lenses separated by arbitrary distances, the amplitude distribution is continuously fractional Fourier transformed as it propagates through the system. The order $\phi(z)$ of the fractional transform observed at the distance z along the optical axis is a monotonically increasing function. The transforms are observed on spherical reference surfaces. Wherever the order of the transform $\phi(z)$ is equal to $(4j + 1)\pi/2$ for any integer j, we observe the Fourier transform of the input. Wherever the order is equal to $(4j + 2)\pi/2$, we observe an inverted image, etc.

The results of this subsection can be generalized for systems composed of an arbitrary sequence of spherical refracting surfaces.

F.   Interpretation of Quadratic Graded-Index Media as a Continuum of Infinitesimal Thin Lenses Interspersed with Infinitesimal Sections of Free Space

The fractional Fourier-transforming property of quadratic graded-index media was discussed in Refs. 4–7 and 10. The refractive index in such a medium has the following dependence on x:

$$n^2(x) = n_0{}^2[1 - (x/\xi)^2], \tag{59}$$

where $n_0 > 0$ and $\xi > 0$ are the medium parameters. In considering such a medium of length l, the output is related to the input by Eq. (34) just as well, provided that we replace $\lambda \to \lambda/n_0$. Now $\phi \equiv a\pi/2$ and $s > 0$ are given in terms of n_0, ξ, and l by

$$\phi = l/\xi, \tag{60}$$

$$s^2 = \lambda\xi/n_0. \tag{61}$$

Now we discuss the relation between such fractional Fourier transformers and the canonical systems discussed in Subsection 3.C. Let us consider Type I (the discussion is similar for Type II). Assume that we cascade N Type I systems, each of length $2d$, so that the overall length of the system is $l = N2d$. Now, keeping this overall length and the scale parameter s fixed, we let $N \to \infty$ and $2d \to 0$. Physically, what we obtain is a large number of closely spaced weak-focal-power lenses.[16] In the limit the average refractive-index distribution of this assembly will have a quadratic dependence on x, similar to Eq. (59).

To see that this is true, let us consider optical paths parallel to the optical axis. Using small ϕ approximations of Eqs. (39) and (40), we can show that the phase collected along such a path through the infinite cascade of infinitesimal Type I systems is

$$\simeq -\pi\phi(x/s)^2, \tag{62}$$

where we have dropped the constant term $2\pi N2d/\lambda$. Turning our attention to quadratic index media, Eqs. (59)–(61) allow us to write the collected phase in exactly the same form as above. This means that if we assume functional equivalence of the two systems (identical values of ϕ and s) we can deduce their physical equivalence (identical collected phase, which is proportional to optical density).

From the same sets of equations we can derive another relation that is valid for both cases:

$$\phi = (\lambda/s^2)\ell, \tag{63}$$

where $\lambda \to \lambda/n_0$ for the quadratic index case.

4. GENERAL CONDITIONS FOR THE OPTICAL FRACTIONAL FOURIER TRANSFORM

The class of quadratic-phase systems[17–21] is characterized by linear transformations of the form

$$p_{\text{out}}(x) = \int_{-\infty}^{\infty} h(x, x') p_{\text{in}}(x') \mathrm{d}x',$$

$$h(x, x') = C \exp[i\pi(\alpha x^2 - 2\beta xx' + \gamma x'^2)], \qquad (64)$$

where C is a complex constant, and α, β, and γ are real constants independent of x and x'. Optical systems involving an arbitrary sequence of thin lenses and sections of free space (in the Fresnel approximation) belong to this class. (We can easily show this by first proving that the kernel for a section of free space followed by a lens and a lens followed by a section of free space both assume the above form and by then noting that an arbitrary system can be composed of these units.)

For this class of systems it is sufficient to specify the parameters C, α, β, and γ to characterize the system completely. The condition for a fractional Fourier transform can be stated in terms of these parameters as follows. There should exist scale parameters s_1, s_2 such that

$$\alpha s_2{}^2 = \cot \phi = \gamma s_1{}^2, \qquad (65)$$

$$\beta s_1 s_2 = \csc \phi \qquad (66)$$

for some ϕ. It can be shown that such scale parameters can be found if and only if $0 \le \alpha\gamma \le \beta^2$. If this condition is satisfied, the necessary scale parameters and the order of the resulting transform are given by

$$s_1{}^4 = (\beta^2\gamma/\alpha - \gamma^2)^{-1}, \qquad (67)$$

$$s_2{}^4 = (\beta^2\alpha/\gamma - \alpha^2)^{-1}, \qquad (68)$$

$$\tan \phi = \pm(\beta^2/\alpha\gamma - 1)^{1/2}, \qquad (69)$$

where the sign of $\tan \phi$ is the same as the identical signs of α and γ and where ϕ lies in the interval $[-\pi, 0]$ or $[0, \pi]$ according to whether $\beta s_1 s_2 \le 0$ or $\beta s_1 s_2 \ge 0$. For example, if $s_1 s_2 \ge 0$, $\alpha \ge 0$, and $\beta \ge 0$, ϕ lies in the interval $[0, \pi/2]$.

If the condition $0 \le \alpha\gamma \le \beta^2$ is not satisfied, it is still possible to observe a fractional Fourier transform between spherical (rather than planar) reference surfaces at the input and the output. (That is, it is possible to take any system and make a fractional Fourier transformer out of it by appending lenses to the input and the output.) To show this, we note that the kernel $h'(x, x')$ between the new spherical reference surfaces will be modified according to $\alpha' = \alpha - 1/\lambda R_2$, $\beta' = \beta$, and $\gamma' = \gamma + 1/\lambda R_1$, where R_1 and R_2 are the radii of the new input and output reference surfaces. Now we can always ensure that $0 \le \alpha'\gamma' \le \beta'^2$ by an appropriate choice of R_1 and R_2.

It is possible to arrive at the same results through a different formalism. Let us define the transformation matrix characterizing the optical system as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv \begin{bmatrix} \gamma/\beta & 1/\beta \\ -\beta + \alpha\gamma/\beta & \alpha/\beta \end{bmatrix}$$

$$= \begin{bmatrix} \alpha/\beta & -1/\beta \\ \beta - \alpha\gamma/\beta & \gamma/\beta \end{bmatrix}^{-1}, \qquad (70)$$

with determinant $AD - BC = 1$. There are many reasons for defining such a matrix. First, if several systems, each characterized by such a matrix, are cascaded, one can find the matrix characterizing the overall system by multiplying the matrices of the several systems. Second, the effect of the optical system on the Wigner distribution[17,18] of the input is easily expressed in terms of this matrix. Third, this matrix can be made the basis of a geometrical-optics description of a quadratic-phase system; it is essentially the well-known ray matrix.[15] These issues are discussed extensively in the research of Bastiaans,[17–21] so they are not considered further here. (Bastiaans actually deals with the inverse of the above matrix.)

Introducing the variables $u \equiv x/s_2$ and $u' \equiv x'/s_1$ in Eqs. (64), one can write the kernel as $\propto \exp[i\pi(\overline{\alpha}u^2 - 2\overline{\beta}uu' + \overline{\gamma}u'^2)]$, with $\overline{\alpha} = \alpha s_2{}^2$, $\overline{\beta} = \beta s_1 s_2$, and $\overline{\gamma} = \gamma s_1{}^2$. It is now possible to show that the elements of the new transformation matrix associated with the dimensionless variables are related to the elements of the original transformation matrix given in Eq. (70):

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} = \begin{bmatrix} \overline{\gamma}/\overline{\beta} & 1/\overline{\beta} \\ -\overline{\beta} + \overline{\alpha}\,\overline{\gamma}/\overline{\beta} & \overline{\alpha}/\overline{\beta} \end{bmatrix}$$

$$= \begin{bmatrix} As_1/s_2 & B/s_1 s_2 \\ Cs_1 s_2 & Ds_2/s_1 \end{bmatrix}. \qquad (71)$$

The transformation matrix associated with the fractional Fourier transform of order ϕ is the rotation matrix[4,9,10]

$$\begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}. \qquad (72)$$

Thus the condition for a fractional Fourier transform is that there exist scale parameters s_1 and s_2 such that the matrices given in expressions (71) and (72) are equal for some ϕ. The necessary and sufficient condition for such scale parameters to exist can be expressed as $0 \le AD \le 1$ (or, equivalently, $-1 \le BC \le 0$). If this condition is satisfied, the necessary scale parameters and the order of the resulting transform are given by

$$s_1{}^4 = B^2(A/D - A^2)^{-1}, \qquad (73)$$

$$s_2{}^4 = B^2(D/A - D^2)^{-1}, \qquad (74)$$

$$\cos \phi = \pm(AD)^{1/2}, \qquad (75)$$

where $\text{sgn}(\cos \phi) = \text{sgn}(As_1/s_2)$ and ϕ lies in the interval $[-\pi, 0]$ or $[0, \pi]$ according to whether $B/s_1 s_2 \le 0$ or $B/s_1 s_2 \ge 0$. For example, if $s_1 s_2 \ge 0$, $A \ge 0$, and $B \ge 0$, ϕ lies in the interval $[0, \pi/2]$.

If the condition $0 \le AD \le 1$ is not satisfied, we can still make a fractional Fourier transformer out of this system by appending lenses at its input and output surfaces. (That is, for any system, it is possible to find spherical input and output reference surfaces between which a fractional Fourier-transform relation exists.) To show this, we multiply the transformation matrix from the left and the right by the transformation matrix of a thin lens[15]:

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -K_2 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -K_1 & 1 \end{bmatrix}, \qquad (76)$$

where K_1, K_2 are measures of the powers of the lenses. Multiplying the matrices, we find that $A' = A - K_1 B$ and $D' = D - K_2 B$. It is clearly possible to satisfy the condition $0 \leq A'D' \leq 1$ by an appropriate choice of K_1 and K_2.

5. CONCLUSIONS

There exists a fractional Fourier-transform relation between two spherical reference surfaces of given radii and separation. It is possible to determine the order and the scale parameters associated with this fractional transform, given the radii and the separation of the surfaces. Alternatively, given the desired order and scale parameters, it is possible to determine the necessary radii and separation.

The propagation of light along the $+z$ direction can be viewed as a process of continual fractional Fourier transformation. As light propagates, its distribution evolves through fractional transforms of increasing orders. The order $\phi(z)$ of the fractional transform observed at z is a continuous monotonic increasing function of z. (The fractional transform at z is observed on a spherical reference surface intersecting the optical axis at that location.) One of the central results of diffraction theory is that the far-field diffraction pattern is the Fourier transform of the diffracting object. Thus we have shown that the field at a closer distance is the fractional Fourier transform of the diffracting object.

The effect of a thin lens on such a propagating waveform is merely to bend the spherical reference surface into another reference surface with a different radius. Thus we can continue tracking the evolution of the wave in terms of fractional Fourier transforms, starting from this new reference surface.

It has been shown by Onural[22] that the propagation of light in the Fresnel approximation can be viewed as a wavelet transform. The relation between fractional Fourier transforms and wavelet transforms is discussed in Ref. 7, so that Onural's result can be tied to ours.

We have discussed rather carefully what we have termed canonical fractional Fourier-transforming systems, which are the simplest symmetric configurations yielding a fractional Fourier transform between planar reference surfaces. We also showed that quadratic graded-index media can be viewed as the limit of a larger and larger number of weaker and weaker lenses.

We derived the (well-known) single-lens imaging equations, starting from an imaging condition stated in terms of fractional Fourier transforms. Generalizing, we saw that fractional Fourier transforms provide a new way of analyzing optical systems involving several lenses separated by arbitrary distances. Such systems can be analyzed by means of geometrical optics, Fresnel integrals (spherical-wave expansions), plane-wave expansions, Hermite–Gaussian beam expansions, and, as we showed, fractional Fourier transforms. The various approaches prove useful in different situations and provide different viewpoints that complement each other. The fractional Fourier-transform approach is appealing in that it describes the continuous evolution of the wave as it propagates through the system.

Finally, we considered a rather general class of optical systems and stated the general conditions that a member of this class must satisfy for its output to be viewed as the fractional Fourier transform of its input. We also showed that one can make a fractional Fourier transformer out of any system by appending lenses of appropriate focal length at the input and the output faces.

It is also of interest to formulate our results in the framework and the conventions of Hermite–Gaussian beams. This approach is suitable for studying beam propagation and spherical mirror resonators.[14]

We have avoided discussing complex-order transforms so as to avoid dealing with complex-valued scale parameters s_1 and s_2. Such systems do not satisfy $0 \leq g_1 g_2 \leq 1$. Thus, if Fig. 1 is interpreted as a spherical mirror resonator, we are dealing with an unstable resonator.[14] This case might be worth investigating.

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Note added in proof: The reader is also referred to Refs. 23 and 24, two recent publications that discuss similar results.

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