The fractional Fourier transform and its applications in optics and signal processing

The fractional Fourier transform (fractional FT) is a generalization of the common Fourier transform with an order parameter a. Mathematically, the ath order fractional FT is the ath power of the fractional FT operator. The $a=1$st order fractional transform is the common Fourier transform. The $a=0$th transform is the function itself. With the development of the fractional FT and related concepts, we see that the common frequency domain is merely a special case of a continuum of fractional domains, and arrive at a richer and more general theory of alternate signal representations, all of which are elegantly related to the notion of space-frequency distributions.

Every property and application of the common Fourier transform becomes a special case of that for the fractional transform. In every area in which Fourier transforms and frequency domain concepts are used there exists the potential for generalization and improvement by using the fractional transform. For instance, the theory of optimal Wiener filtering in the common Fourier domain can be generalized to optimal filtering in fractional domains, resulting in smaller mean square errors at practically no additional cost. The well-known result stating that the far-field diffraction pattern of an aperture is in the form of the FT of the aperture can be generalized to state that, at closer distances, one observes the fractional FT of the aperture.

Applications

The fractional FT has been found to have several applications in the area known as analog optical information processing or Fourier optics. This transform allows a reformulation of this area in a much more general way than the standard formulation. It has also allowed a generalization of the Fourier transform and the notion of the frequency domain: very central concepts in signal processing. It is therefore expected to have an impact in the form of deeper understanding or new applications in every area in which the Fourier transform plays a significant role.

More specifically, some applications which have already been investigated or suggested include Fresnel diffraction, optical beam propagation and spherical mirror resonators (lasers), propagation in graded index media, optical systems design, quantum optics (squeezed states), phase retrieval, signal detection, pattern recognition, radar, tomography, noise representation, time-variant filtering and multiplexing, data compression, study of space/time-frequency distributions, and solution of differential equations. We believe that these are only a fraction of the possible applications.

Optical implementation of the fractional FT

The fractional Fourier transform can be optically implemented in a manner similar to that of the common Fourier transform (Mendlovic and Ozaktas, Lohmann). The fact that the fractional FT can be realized optically means that the many applications of the transform in signal processing can also be carried over to optical signal processing.

The fractional FT as a tool for analyzing optical systems

It has been shown that there exists a fractional FT relation between the (appropriately scaled) optical amplitude distributions on two spherical reference surfaces with given radii and separation (Ozaktas and Mendlovic, Pellat-Finet and Bonnet). This result provides an alternative statement of the law of propagation and allows us to pose the fractional FT as a tool for analyzing and describing a rather general class of optical systems. One of the central results of diffraction theory is that the far-field diffraction pattern is the FT of the diffracting object. It is possible to generalize this result by showing that the field patterns at closer distances are the fractional FTs of the diffracting object.

More generally, in an optical system involving many lenses separated by arbitrary distances, it is possible to show that the amplitude distribution is continuously fractional Fourier transformed as it propagates through the system. The order $a(z)$ of the fractional transform observed at the distance z along the optical axis is a continuous monotonic increasing function. As light propagates, its distribution evolves through fractional transforms of increasing orders. Wherever the order of the transform $a(z)$ is equal to $4j+1$ for any integer j, we observe the Fourier transform of the input. Wherever the order is equal to $4j+2$, we observe an inverted image, etc.

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(Those wishing to learn more may consult these references and the references therein.)

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