Invited Paper

Fractional Fourier Transform in Optics

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Abstract:

The Fourier transform is one of the most important concepts in optical signal processing, and in the mathematical theory of signals and systems. It can be implemented optically as well as digitally, and constitutes the building block of more complex information processing operations. The fractional Fourier transform is a generalization of the ordinary Fourier transform and also has efficient digital and optical implementations. However, the fractional order parameter associated with the fractional Fourier transform makes it more versatile than the ordinary Fourier transform, leading to many applications in optics and signal processing.

The fractional Fourier transform is a linear integral transformation which is a generalization of the ordinary Fourier transform. The fractional Fourier transform operator F^a has an order parameter a. Mathematically, the a^{th} order fractional Fourier transform is the a^{th} power of the ordinary Fourier transform operator F. The $a=1^{st}$ order fractional transform is the ordinary Fourier transform. The $a=0^{th}$ transform is the function itself. With the development of the fractional Fourier transform and related concepts, we see that the ordinary frequency domain is merely a special case of a continuum of fractional domains, and arrive at a richer and more general theory of alternate signal representations. These representations are elegantly related to the notion of space-frequency (phase space) distributions. Every property and application of the ordinary Fourier transform becomes a special case for the fractional transform. In every area in which Fourier transforms and frequency domain concepts are used, there exists the potential for generalization and improvement by using the fractional transform.

The most representative example is the translation of the correlation/convolution operations to fractional correlation/convolution. This provides unique capabilities to perform space variant filtering tasks. A more particular example is the theory of optimal Wiener filtering in the ordinary Fourier domain. By considering optimal filtering in fractional domains, one can obtain smaller mean square errors at practically no additional cost.

In optics, the well-known result stating that the far-field diffraction pattern of an aperture is in the form of the Fourier transform of the aperture can be generalized to state that at closer distances, one observes the fractional Fourier transform of the aperture. It has been shown that wave propagation phenomena can be modelled and expressed in terms of the fractional Fourier transform, and that the laws of propagation can be stated as a process of continual fractional Fourier transformation. Extensions to this theory allow us to pose the fractional Fourier transform as a tool for analyzing and describing a rather general class of optical systems.

More generally, in an optical system involving many lenses separated by arbitrary distances, it is possible to show that the amplitude distribution is continuously fractional Fourier transformed as it propagates through the system. The order $a(z)$ of the fractional transform observed at the distance z along the optical axis is a continuous monotonic increasing function. As light propagates, its distribution evolves through fractional

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transforms of increasing orders. Wherever the order of the transform $a(z)$ is equal to $4k+1$ for any integer k, we observe the Fourier transform of the input. Wherever the order is equal to $4k+2$, we observe an inverted image, etc. Propagation in graded-index media, and Gaussian beam propagation have also been studied in terms of the fractional Fourier transform.

When we had reviewed the fractional Fourier transform and its applications three years ago in [1], it was possible to include a list of almost all recent publications in the area. Now, space does not permit us to give even a representative sample of the hundreds of publications which have appeared on the subject. Three recent reviews contain references to the more important works [2,3,4].

The fractional Fourier transform allows a reformulation of the area known as analogue optical information processing or Fourier optics, in a much more general way than the standard formulation. As mentioned above, the fractional Fourier transform is a useful tool for analyzing and describing a rather general class of optical systems.

Some applications of the transform in digital and optical signal and image processing which have already been investigated or suggested include time- or space-variant filtering, estimation, restoration, phase retrieval, reconstruction, detection, multiplexing, data compression, correlation, matched filtering, pattern recognition, and linear system synthesis. The transform has been found useful for the study of time- or space-frequency distributions, for signal synthesis, for radar problems, and for the solution of differential equations. The relationship to wavelet transforms and neural networks has been explored and other fractional transformations have been inspired. The discrete-time fractional Fourier transform and its digital computation have also been investigated.

The fractional Fourier transform can be optically realized in a similar manner as the common Fourier transform. The fact that the fractional Fourier transform can be realized optically means that the many applications of the transform in signal processing can also be carried over to optical signal processing.

Other optical applications include spherical mirror resonators (lasers), optical systems and lens design, quantum optics, phase retrieval, quantum state reconstruction, statistical optics, beam shaping and wavefield synthesis, and Legendre transformations.

References

[1] H. M. Ozaktas and D. Mendlovic, "Applications of the fractional Fourier transform in optics and signal processing--a review," *17th Congress of the International Commission for Optics: Optics for Science and New Technology* SPIE Proceedings **2778,** 414--417. SPIE, Bellingham, Washington, (1996)

[2] H. M. Ozaktas, M. A. Kutay, and D. Mendlovic, "Introduction to the fractional Fourier transform and its applications," P. W. Hawkes, editor, *Advances in Imaging and Electron Physics*, Volume **106**, 239--291. Academic Press, San Diego, California, (1998)

[3] D. Mendlovic, Z. Zalevsky, and H. M. Ozaktas, "Applications of the fractional Fourier transform to optical pattern recognition," F. T. S. Yu and S. Jutamulia, editors, *Optical Pattern Recognition*, 89--125. Cambridge University Press, Cambridge, (1998)

[4] A. W. Lohmann, D. Mendlovic and Z. Zalevsky, "Fractional transformations in optics," E. Wolf editor, *Progress In Optics XXXVIII*, 263-342 (1998)