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Relationship between the beam propagation method and linear canonical and fractional Fourier transforms

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The beam propagation method (BPM) can be viewed as a chain of alternating convolutions and multiplications, as filtering operations alternately in the space and frequency domains or as multiplication operations sandwiched between linear canonical or fractional Fourier transforms. These structures provide alternative models of inhomogeneous media and potentially allow mathematical tools and algorithms associated with these transforms to be applied to the BPM. As an example, in the case where quadratic approximation is possible, it is shown that the BPM can be represented as a single LCT system, leading to significantly faster computation of the output field.

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1. INTRODUCTION

The beam propagation method (BPM) is an important computational tool for solving the time-harmonic Helmholtz equation under the slowly varying envelope approximation (SVEA) [1–4]. It can be used for beam propagation calculations in inhomogeneous media in which refractive index changes are small relative to the average index, such that the SVEA can hold. It is mostly used to simulate and study optical waveguides and other optical devices with inhomogeneous refractive index distributions. It has also been modified for use in analysis of diffraction gratings and for use in anisotropic media [5,6].

Quadratic-phase systems (QPSs), mathematically known as linear canonical transforms (LCTs) [7,8], can model a broad class of optical systems including thin lenses, sections of free space in the Fresnel approximation, sections of quadratic graded-index (GRIN) media, and arbitrary concatenations of any number of these. Such concatenations of these basic components are sometimes referred to as first-order optical systems [9–13]. Fractional Fourier transforms (FRTs) [14,15], scaling operations, and chirp multiplication (CM) and chirp convolution (CC) operations—the latter also known as the Fresnel transform—are special cases of QPSs [9].

Arbitrary QPSs or LCTs, no matter how many lenses and sections of free space they are composed of, can be decomposed into a sequence of three or four elementary operations, such as the forms CC-CM-CC or CM-CC-CM, or other sequences involving FRTs and scaling [9]. The term Fourier-optical systems (FOSs) refers to the class of systems consisting of multiplicative spatial filters sandwiched between QPSs. FOSs include arbitrary sequences of lenses, free space, spatial filters, and quadratic GRIN media. It has been shown that arbitrary FOSs can be expressed as multiplicative spatial filters sandwiched between FRT stages [16]; therefore, such systems can be modeled as multi-stage fractional Fourier domain filtering systems [17–20], where several multiplicative filters are applied in consecutive FRT domains. Furthermore, it has been shown that such consecutive FRT filtering operations are equivalent to the application of spatial filters alternately in the space and frequency domains [21]. Therefore, FOSs can be modeled as repeated filtering alternately in the space and frequency domains, which is also equivalent to an alternating chain of convolution and multiplication operations.

QPSs or LCTs are capable of exactly representing quadratic GRIN media [22], in which the inhomogeneous refractive index distribution is given by $n^2(x) = n_1^2[1 - (n_2/n_1)x^2]$, where n_1 and n_2 are the medium parameters and x is the transverse coordinate. The parameters n_1 and n_2 are assumed to be constants along the propagation direction.

Thus, any FOS can be represented in any of the following equivalent forms:

1. Multiplicative filters sandwiched between LCTs.
2. Multiplicative filters sandwiched between FRTs.
3. Multiplicative filters sandwiched between FTs.
4. Alternating multiplications and convolutions.
5. Multiplicative filters sandwiched between Fresnel transforms (sections of free space or other homogeneous media).

In this paper, we show that the BPM method can be viewed as the alternate application of any of the following: multiplications and LCTs/QPSs, multiplications and FRTs, multiplications and FTs, multiplications and convolutions, or multiplications and homogeneous propagation. This is also consistent with the view of modeling an inhomogeneous medium such as the cascade of a large number of spatial filters and narrow sections of propagation. The narrow sections of propagation can be represented as QPSs/LCTs or FRTs, in addition to the traditional choice of Fresnel transforms.

We will further show that, under certain circumstances, each layer in the BPM can be approximately represented by a cascade of two fundamental operations: CC and CM. The former CC step corresponds to Fresnel propagation in a homogeneous medium, and the latter CM step corresponds to the approximated phase correction. When this approximation is acceptable, the CC and CM operations for all layers can be collapsed into a single equivalent QPS, which can be computed for about the same cost as an ordinary Fourier transform. This leads to very substantial computational savings.

The paper is organized as follows. In Sections 2 and 3, we summarize the basics of the BPM and QPS, respectively. In Section 4, we derive the relation between the BPM and QPSs. In Section 5, we present examples of our method, and Section 6 is devoted to concluding remarks.

2. BASICS OF THE BEAM PROPAGATION METHOD

In this section, we summarize the basic structure of the BPM. We consider the 2D case with propagation in the z direction and with x representing the transverse direction. We have an inhomogeneous medium with refractive index distribution $n(x, z)$. We consider a time-harmonic monochromatic wave field $E(x, z, t) = \mathbf{Re}\{U(x, z)e^{-j\omega t}\}$ that propagates in this inhomogeneous medium. $U(x, z)$ is the complex amplitude, and ω is the angular frequency. The scalar wave equation is given by

$$\frac{\partial^2 E}{\partial t^2} = \frac{c^2}{n^2}\nabla^2 E, \tag{1}$$

where c is the speed of light and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$. By using the time-harmonic assumption, the wave equation is simplified to the Helmholtz equation in an inhomogeneous medium:

$$(\nabla^2 + n^2 k_0^2)U(x, z) = 0, \tag{2}$$

where $k_0 = 2\pi/\lambda_0$ and λ_0 is the free-space wavelength.

The first key assumption of the BPM is that we have a refractive index distribution of the form $n(x, z) = \bar{n} + \Delta n(x, z)$, where $\Delta n \ll n$, meaning that the inhomogeneous medium is such that the refractive index varies within a small neighborhood of an average value $\bar{n}$ (weak index modulation). Additionally, when we make the paraxial approximation and assume that the wave propagation is along directions making very small angles with the z direction, $U(x, z)$ can be written as

$$U(x, z) = \bar{U}(x, z)e^{-j\bar{n}k_0 z}, \tag{3}$$

where $\bar{U}(x, z)$ is a slowly varying function of x and z. The slowly varying envelope approximation is justified by the assumptions of weak index modulation and paraxial propagation. By substituting Eq. (3) into Eq. (2), we get

$$\left(\nabla^2 - 2j\bar{n}k_0\frac{\partial}{\partial z} + n^2 - \bar{n}^2\right)\bar{U} = 0. \tag{4}$$

The SVEA allows us to neglect the second-order partial derivative of $\bar{U}$ with respect to z in Eq. (4), leading to the paraxial wave equation for inhomogeneous media [1],

$$\frac{\partial}{\partial z}\bar{U} = \frac{-j}{2\bar{n}k_0}\left[\frac{\partial^2\bar{U}}{\partial x^2} + (n^2 - \bar{n}^2)k_0\bar{U}\right]. \tag{5}$$

Equation (5), which is fundamental to the BPM, can be solved by an iterative approach. To implement Eq. (5), we can divide the z axis into N slices of length Δz and treat each slice as a combination of (i) propagation over a distance Δz in a homogeneous media of refractive index $\bar{n}_i$, where $i = 1, 2, \ldots, N$ denote the average refractive index value along the corresponding slice, and (ii) passage through a phase filter. The phase filter $\exp(-j(n(x, z_i) - \bar{n}_i)\Delta z)$ accounts for the slowly varying index distribution, which models the inhomogeneous media. Note that $z_i = i\Delta z$ and z_0 is the input plane. Starting from an initial wave field at $z = 0$, the BPM method iterates this initial value through each slice, using the output of the ith slice as the input of the $(i+1)$th slice. By letting N be large and Δz be small, the assumptions in the derivation of Eq. (5) are justified. In this manner, the slowly varying inhomogeneous index variation is separated from the homogeneous propagation by virtue of Eq. (5), and after iterating through all the slices, one can obtain the output wave field.

For the homogeneous propagation step, the method most commonly used is the angular spectrum method (ASM) that relies on the angular spectrum of planes waves [1,3,4,23]. If ASM is used to implement the homogeneous propagation part, one can write the iteration equation for one slice between z_i and z_{i+1} as

$$U(x, z_{i+1}) = \mathcal{F}^{-1}\left\{\mathcal{F}\left(U(x, z_i)\exp\left(j2\pi\Delta z\sqrt{\frac{\bar{n}_i^2}{\lambda_0^2} - f_x^2}\right)\right)\right.$$
$$\left.\times\exp\left(-j(n(x, z_i) - \bar{n}_i)k_0\Delta z\right)\right\}, \tag{6}$$

where $z_{i+1} = z_i + \Delta z$ and $\bar{n}_i$ is the average refractive index for the ith slice. $\mathcal{F}$ stands for the Fourier transformation, and f_x is the spatial frequency associated with the variable x. Also note that Eq. (6) assumes $f_x < \bar{n}_i/\lambda_0$, which means that evanescent waves are excluded.

We can write this in operator notation for later use as

$$U(x, z_{i+1}) = \mathcal{C}_i\,\mathcal{P}_{\bar{n}_i}\,U(x, z_i), \tag{7}$$

where $\mathcal{P}_{\bar{n}_i}$ stands for free-space propagation in a homogeneous region of length Δz and refractive index $\bar{n}_i$, and where $\mathcal{C}_i$ is the operator representing the multiplicative phase compensation $C_i = \exp(-j(n(x, z_i) - \bar{n}_i)k_0\Delta z)$ for the ith slice.

The advantages of BPM are that it is relatively simple, straightforward, and fast. More advanced BPM algorithms have been developed to work in the non-paraxial case as well. For example, [1,4] employ the ASM to calculate the propagation part. This allows use of the fast Fourier transform (FFT) and, thus, has a computational advantage. The version of BPM that we consider is that given in [3,24]. While we consider the case where we have only transverse dimension, generalization to two transverse dimensions is not difficult.

3. QUADRATIC-PHASE SYSTEMS

A QPS, mathematically known as a LCT, is a unitary system with parameter matrix $\mathbf{M}$, whose output $f_{\mathbf{M}}(u)$ is related to its input $f(u)$ through a quadratic-phase integral:

$$f_{\mathbf{M}}(u) = \sqrt{\beta} e^{-j\pi/4} \int_{-\infty}^{\infty} \exp\left[j\pi(\alpha u^2 - 2\beta u u' + \gamma u'^2)\right] \times f(u')\mathrm{d}u', \tag{8}$$

where α, β, and γ are real parameters. Systems or transforms characterized by this type of relationship are also known under other names including ABCD systems and, along with the FRTs that they generalize, have received considerable attention and found many applications [9,10,25–56].

The 2×2 matrix $\mathbf{M}$ whose elements are A, B, C, D represents the same information as the three parameters α, β, and γ, which uniquely define the QPS:

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \gamma/\beta & 1/\beta \\ -\beta + \alpha\gamma/\beta & \alpha/\beta \end{bmatrix} = \begin{bmatrix} \alpha/\beta & -1/\beta \\ \beta - \alpha\gamma/\beta & \gamma/\beta \end{bmatrix}^{-1}. \tag{9}$$

The unit-determinant matrix $\mathbf{M}$ belongs to the class of unimodular matrices. More on the group-theoretical structure of QPSs may be found in [9,25].

The result of repeated application (concatenation) of QPSs can be handled easily with the above-defined matrix. When two or more QPSs are cascaded, the resulting system is again a QPS whose matrix is obtained by multiplying the matrix of each QPS in the cascade structure. That is, if two QPSs with matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ operate in a successive manner, then the equivalent system is a QPS with the matrix $\mathbf{M}_3 = \mathbf{M}_2\mathbf{M}_1$. QPSs are not commutative. The matrix of the inverse of an QPS is simply another QPS whose matrix is the inverse of the matrix of the original QPS [9,25].

4. RELATIONSHIP BETWEEN THE BEAM PROPAGATION METHOD AND QUADRATIC-PHASE SYSTEMS

As discussed in Section 2, the BPM involves alternating application of two basic operations. The first is a propagation calculation, which may be based on the ASM but also corresponds to a convolution. The second part is application of a phase filter, which corresponds to a multiplication. That is, the BPM involves a sequence of alternating convolutions and multiplications. Since convolution corresponds to multiplication in the Fourier domain and vice versa, the propagation part can be viewed as a multiplicative filter in the Fourier domain. Thus, we can also say that BPM corresponds to the consecutive application of a multiplicative filter (or a convolution) alternately in the space and spatial frequency domains.

Thus, we can state our first main result. The BPM method can be viewed as the alternate application of:

1. Multiplications and LCTs/QPSs.
2. Multiplications and FRTs.
3. Multiplications and FTs.
4. Multiplications and convolutions.
5. Multiplications and homogeneous propagation.

The last three follow from our discussion at the beginning of this section. The first two follow from the equivalences discussed in Section 1, which are based on [16,21].

The above result is also consistent with the modeling of an inhomogeneous medium as a large number of spatial filters alternating with narrow sections of propagation. The latter can be represented as QPSs/LCTs or FRTs, in addition to the conventional choice of Fresnel transforms. In the limit that these sections become narrower and narrower, this model becomes a mathematically exact one.

FRTs and LCTs have been extensively studied and have a large number of properties that are generalizations of the properties of ordinary Fourier transforms [9]. Interpreting the BPM method in terms of FRTs and LCTs opens up the possibility of exploiting these properties in the study of BPM.

An important observation is that such a chain of operations is in general irreducible. Two (or any number of) multiplication operations can be combined into a single multiplication operation, and two (or any number of) convolution operations can be combined into a single convolution operation. In other words, multiplication or convolution operations, separately, can be collapsed down to a single equivalent operation. Had it been possible to exchange the order of a multiplication and a convolution operation, possibly by using new multiplying and convolving functions, an alternating cascade of convolution-multiplication-convolution-multiplication could be rearranged, merged, and collapsed, simplifying the whole system down to a single multiplication-convolution pair. This would be achieved by switching the order of multiplication and convolution operations such that several multiplications are made adjacent to each other and several convolutions are made adjacent to each other, after which these could be collapsed to a single multiplication and convolution, respectively. That this is not in general possible can be stated as a mathematical result [9]:

Let $g(x) = h_2(x) * [h_1(x) f(x)]$, where $f(x)$ is the input, $g(x)$ is the output, and $h_1(x)$, $h_2(x)$ are given functions. Then it is not in general possible to find $h'_1(x)$, $h'_2(x)$ such that $g(x) = h'_2(x)[h'_1(x) * f(x)]$ defines the same input–output relation between $f(x)$ and $g(x)$ as before.

This result means that alternating multiplication-convolution chains, or equivalently alternating space-frequency filtering chains, cannot be reduced to a smaller number of operations.

Typically, BPM is implemented with a large number of slices to better satisfy the assumptions made in its derivation. Although faster than many other computational approaches, it still carries a significant computational burden due to repetitive calculations on the usually large number of data points of the wave function, which is a consequence of the above-discussed irreducibility. The large number of multiplications, convolutions, or Fourier transforms cannot be combined and collapsed; they must be repeated alternately.

We now show that there is an approximation, if applicable, that allows the whole computation over all the slices to be collapsed down to a single fast calculation. More specifically, when a quadratic-phase approximation is possible, each step of the BPM can be written as a QPS. In this case, by representing each $\mathcal{P}_{\bar{n}_i}$ and $\mathcal{C}_i$ as a QPS and then combining them by using the concatenation rule for QPSs given in Section 3, we can obtain a single overall QPS representing the whole system. This single overall QPS corresponds to the full BPM procedure and is defined by a single ABCD matrix corresponding to a single LCT, as defined by Eq. (8). This is possible because the combination of any number of QPSs is a single QPS, with its parameters determined by combining the parameters of the constituent QPSs. On the other hand, when the quadratic approximation is not possible, such a reduction does not exist.

There exist fast algorithms to digitally calculate QPSs (LCTs) in $\sim M \log M$ time (where M is the number of samples) [36,43,57–61], which are in turn based on fast algorithms for the FRT [62]. Thus, once the system has been modeled, the whole procedure of calculating the output field from the input field can be completed in about the same time it takes to calculate a single Fourier transform. Thus, the iterative nature of the BPM can be bypassed, and computational savings by a large factor can be achieved.

The homogeneous propagation part of the BPM, $\mathcal{P}_{\bar{n}_i}$, can often be implemented by using the Fresnel approximation [23]. The Fresnel transform gives the output field $E(x, z)$ in terms of the input field $E(x, 0)$, after propagation over a distance z in a homogeneous medium of refractive index n:

$$E(x, z) = \frac{e^{jn\frac{2\pi}{\lambda_0}z}}{\sqrt{j\lambda z}} \int E(x', 0) e^{jn\pi\lambda_0 z(x-x')^2} \mathrm{d}x'. \quad \textbf{(10)}$$

This operation is a CC. It is a special case of QPSs, and the corresponding ABCD parameter matrix, denoted by $\mathbf{M}_F$, is given by

$$\mathbf{M}_F = \begin{bmatrix} 1 & \frac{\lambda_0 z}{n} \\ 0 & 1 \end{bmatrix}, \quad \textbf{(11)}$$

where $\alpha = \beta = \gamma = \frac{n}{\lambda_0 z}$.

Now we turn our attention to the multiplicative phase filter $\mathcal{C}_i$. If we can approximate this filter by a quadratic-phase filter, then this operation becomes a CM, which is also a special case of QPSs. This operation can be expressed as $E'(x, z) = e^{-i\pi q x^2} E(x, z)$, and the corresponding ABCD parameter matrix is

$$\mathbf{M}_C = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix}. \quad \textbf{(12)}$$

We will consider a minimum mean square error (MSE) approach to find the best quadratic-phase fit of the form $\exp(-j(ax^2 + b)k_0\Delta z)$ to the actual phase filter $C_i = \exp(-j(n(x, z_i) - \bar{n}_i)k_0\Delta z) = \exp(-j\Delta n(x, z_i)k_0\Delta z)$ in Eq. (6). To find the best fit, we must solve the optimization problem

$$\min_{a_i, b_i} (a_i x^2 + b_i - \Delta n(x, z_i))^2 \quad \textbf{(13)}$$

to obtain the a_i and b_i parameters appearing in the quadratic approximation for each layer, where $(i = 1, 2, \ldots, N)$ represents the BPM slices along z. Numerical solution of the above optimization problem can be carried out by discretizing Eq. (13):

$$\min_{a_i, b_i} \sum_{k=1}^{N_x} (a_i x^2 + b_i - \Delta n(x, z_i))^2, \quad \textbf{(14)}$$

where $(k = 1, 2, \ldots, N_x)$ represents the samples taken along the transverse direction. Solving this optimization problem yields the following 2D system for each i:

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{N_x} x_k^4 & \sum_{k=1}^{N_x} x_k^2 \\ \sum_{k=1}^{N_x} x_k^2 & N_x \end{bmatrix}^{-1} \begin{bmatrix} \sum_{k=1}^{N_x} x_k^2 \Delta n(x_k, z_i) \\ \sum_{k=1}^{N_x} \Delta n(x_k, z_i) \end{bmatrix}. \quad \textbf{(15)}$$

Once the optimization is completed, we can write the ABCD matrix representation $\mathbf{S}_i$ of a single BPM iteration step as follows by multiplying the matrices corresponding to the propagation and multiplicative filter parts:

$$\mathbf{S}_i = \begin{bmatrix} 1 & 0 \\ -2a_i\Delta z/\lambda_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda_0\Delta z/\bar{n}_i \\ 0 & 1 \end{bmatrix}, \quad \textbf{(16)}$$

with the constant phase residue $\exp(j\frac{2\pi}{\lambda_0}\Delta z(\bar{n}_i - b_i))$. Note that the b_i term in this constant phase residue is due to the part of the quadratic approximation and the $\bar{n}_i$ term is due to the constant phase compensation of the Fresnel transform. The above represents once slice of the system. Then, the overall ABCD matrix that approximates the inhomogeneous media with a QPS can be obtained by simple matrix multiplication, requiring manipulation of only 2×2 matrices. The overall matrix $\mathbf{M}_{\text{overall}}$ is given by multiplying the matrices corresponding the many slices:

$$\mathbf{M}_{\text{overall}} = \mathbf{S}_N \mathbf{S}_{N-1} \cdots \mathbf{S}_2 \mathbf{S}_1, \quad \textbf{(17)}$$

and the constant phase residues can be combined at the very end altogether as $\exp(j\frac{2\pi}{\lambda_0}\Delta z \sum_i(\bar{n}_i - b_i))$. Once $\mathbf{M}_{\text{overall}}$ is calculated, we have obtained the ABCD parameters of the *single* overall quadratic-phase transform and, hence, obtained a direct relationship between the input and output. The output can now be computed in terms of the input by fast $\sim M \log M$ algorithms [57,58,61]. Moreover, the cost of optimization, which only includes a matrix multiplication and summations, and the cost of calculating the 2×2 matrix $\mathbf{M}_{\text{overall}}$ do not constitute a significant overhead.

A satisfactory quadratic-phase approximation to the multiplicative phase filters will not always be possible, but when it is, very significant computational gains are achieved.

5. NUMERICAL TESTS

Here we consider some example systems and simulate them using both the BPM and direct fast computation following QPS approximation. This not only serves as a verification of the equivalences presented in Section 4 but also illustrates the speed improvement possible when the quadratic approximation is possible.

In our first example, we consider a medium with an irregular refractive index distribution (Fig. 1). The medium is divided into $N = 25$ layers of width Δz along the propagation direction. Each layer has a different baseline refractive index $\bar{n}_i$ (where $i = 1, 2, \ldots, N$) chosen randomly between 2.5 and 3.5. Each layer is further divided into four subregions with arbitrarily chosen boundary positions and transverse widths as shown in Fig. 1. The refractive index in each subregion is determined by adding to the baseline index, a uniform random deviation with peak value of $\pm 2.5 \times 10^{-3}$. These deviations also vary from layer to layer. As $\Delta n \ll \bar{n}$, we are ensured of the validity of the SVEA. A centered rectangular field with unit amplitude and base width of 2 mm is input to the system and propagates along the z axis for 0.1 mm. The wavelength is 650 nm. The input field is sampled with 256 samples, and 10,000 BPMs are used in the BPM computation. We also approximate this irregularly shaped index distribution with a QPS and directly compute the output with a single quadratic-phase integral. The error between the output fields obtained with conventional BPM and our QPS-based method is only 0.9%. The error is defined as the energy of the difference normalized by the energy of the reference (conventional BPM), expressed as a percentage. In addition to this quite high accuracy, our QPS-based method is 72 times faster than the conventional BPM. The computational times have been measured using MATLAB's *tic* and *toc* methods. The fields at the output of the system are plotted in Figs. 2 and 3.

As a second example, we consider a generalized GRIN system. We assume a standard quadratic GRIN distribution along the transverse direction, but we allow the parameters of the distribution to be different in each slice, which we assume are of 10 nm thickness. The index distribution of this medium is given in Fig. 4, and the average index distribution of this profile along the z direction is plotted in Fig. 5. The parameters in each slice were chosen randomly to generate this example distribution. The wavelength of the light, the sampling parameters, and the input function are the same as in our previous example.

The error between the two methods for this example is 1.64%, and the speed improvement is a factor of 74. The results are plotted in Figs. 6 and 7. This second example shows that the QPS method can be applied to generalized quadratic GRIN

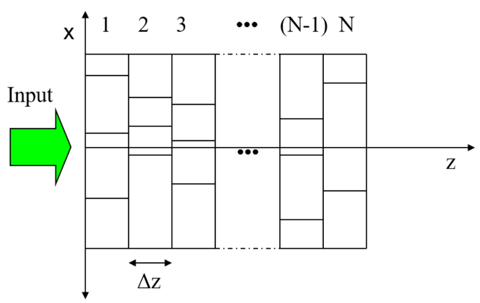

Fig. 1. Test System 1. The direction of propagation is z.

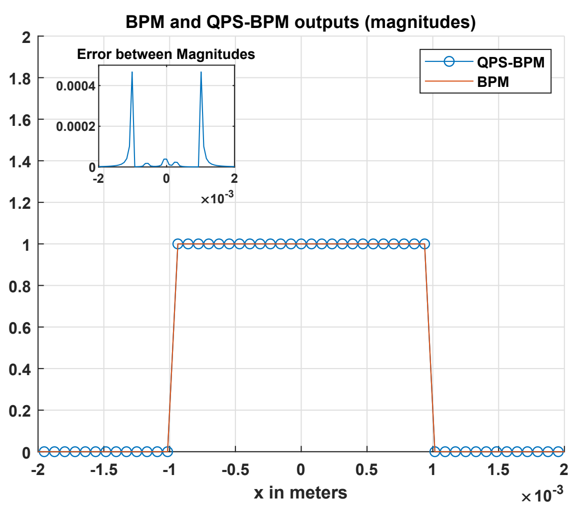

Fig. 2. Output amplitudes for Test System 1.

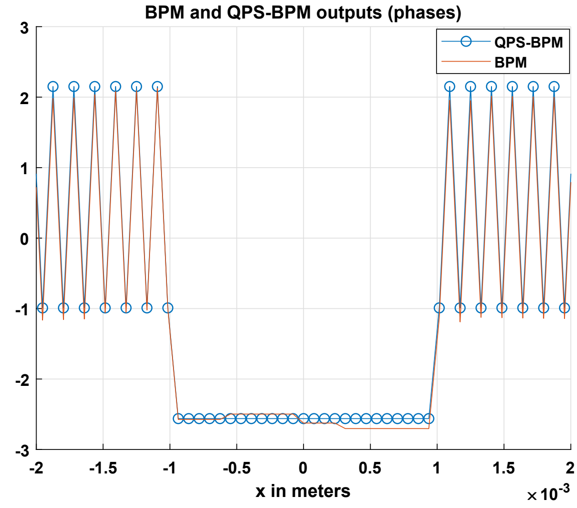

Fig. 3. Output phases for Test System 1.

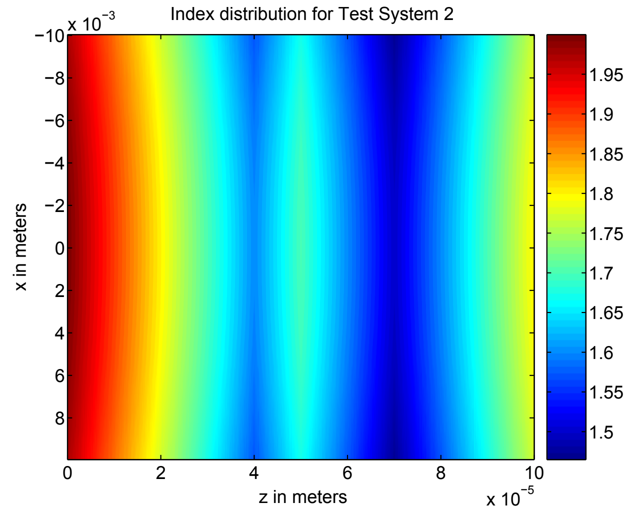

Fig. 4. Index distribution for Test System 2.

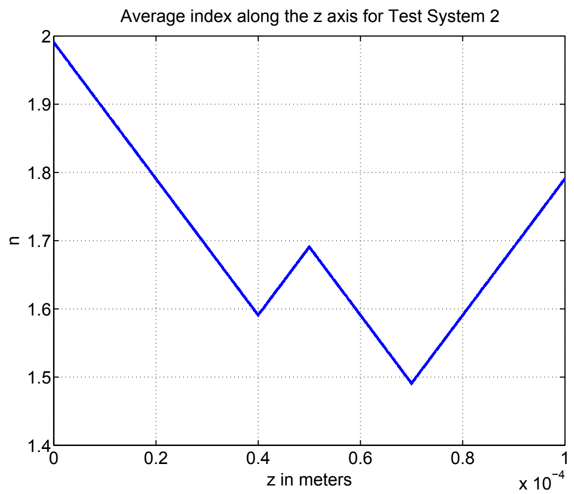

Fig. 5. Average index for Test System 2.

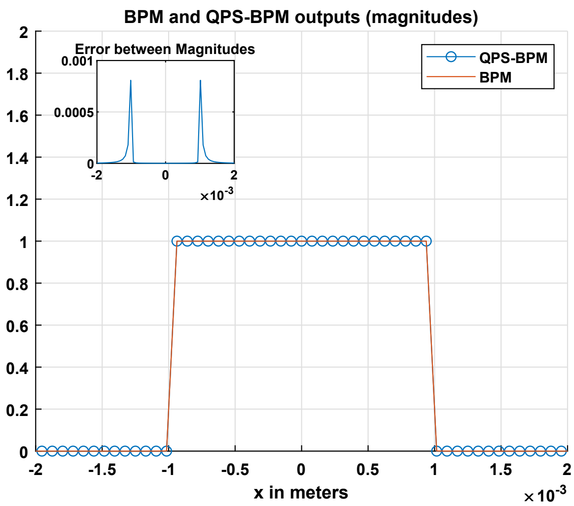

Fig. 6. Output amplitudes for Test System 2.

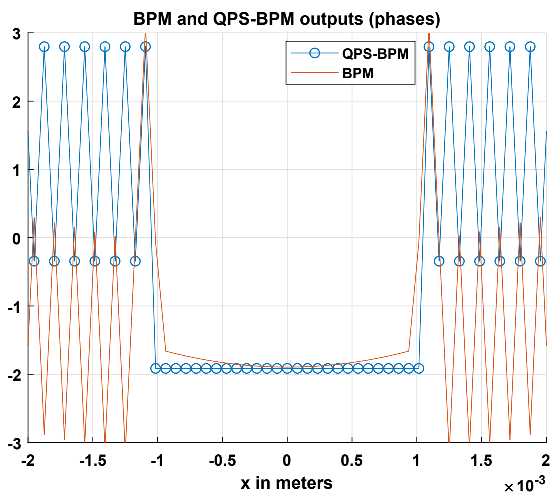

Fig. 7. Output phases for Test System 2.

media whose parameters vary along z. Note that, in both examples, the main contribution to the error comes from the edges, with much better accuracy in the central regions. It should also be noted that whether a given accuracy is considered acceptable or not depends on the application and the demands of the situation. However, given the large speed-ups possible, there could be many situations where the quadratic approximation represents a justified trade-off in terms of speed versus accuracy.

6. CONCLUSION

The BPM can be viewed as a method of numerical analysis. We have seen that the BPM can also be viewed from a signal processing perspective. It can be interpreted as a chain of alternating convolutions and multiplications, as multi-stage filtering alternately in the space and frequency domains, or as multiplication operations sandwiched between QPS/LCT or FRT stages. These mathematical structures can also be viewed as alternative models of inhomogeneous media. The alternative perspectives may allow the application of tools, techniques, and fast algorithms from signal processing to enhance the BPM.

We gave one such example, corresponding to the case where the multiplicative spatial filters can be approximated by quadratic-phase functions. In this case, direct and highly accurate fast computation of the output field from the input field in $\sim M \log M$ time becomes possible—in other words, about the same time it takes to compute a single Fourier transform. In our examples, the speed-up was nearly 2 orders of magnitude. The larger the number of slices that must be used in the BPM, the larger the speed-up obtained by using our direct method is. While we cannot expect the quadratic-phase approximation we employ to be always satisfactory, practical cases where it is sufficient do occur, allowing significant speed improvements.

Further study may lead to a better characterization of the class of systems, which can be sufficiently approximated in this manner. In those cases where the quadratic-phase approximation is not sufficient throughout the system, it may still be possible to identify certain regions where it exhibits sufficient accuracy. Then our direct fast computation method may be used for these regions, and the conventional BPM can be used for the remaining regions. Using this hybrid method instead of using the pure BPM for the entire medium, considerable speed-up may still be possible.

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

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