Time: 9:00-11:15

STUDENT NO:..

Math 206 Complex Calculus – Final Exam

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet.
No correct answer without a satisfying reasoning is accepted. Show your work in detail.
Write your name on the top of every page.

Q-1) Determine the solution $y(n)$ of the difference equation

$$y(n+2) - 6y(n+1) + 9y(n) = 3^n;\ y(0) = 1,\ y(1) = 6,$$

for the given initial conditions. *Note: Use the method of residues for computing the inverse Z-transform.*

Solution: Taking the Z-transform with nonzero initial conditions and solving for $Y(z)$, we get

$$Y(z) = \frac{z(z^2 - 3z + 1)}{(z-3)^3}$$

so that

$$y(n) = \mathrm{R}es_{z=3}[\frac{z^n(z^2 - 3z + 1)}{(z-3)^3}].$$

Since $z = 3$ is a pole of order 3, $y(n) = \frac{1}{2}\phi''(3)$, where $\phi(z) = z^n(z^2 - 3z + 1)$. Taking the derivative twice, we get

$$\phi''(z) = (z^2 - 3z + 1)n(n-1)z^{n-2} + 2(2z-3)nz^{n-1} + 2z^n$$

so that, substituting $z = 3$, dividing by 2, and organizing, we find

$$y(n) = \frac{1}{2}[(n^2 + 17n + 18)3^{n-2}],\quad n \geq 0.$$

Q-2) Solve the initial condition problem

$$\frac{d^3 f}{dt^3} - 3\frac{d^2 f}{dt^2} + 3\frac{df}{dt} - f(t) = H(t-\tau); \quad f''(0) = 1,\ f'(0) = 0,\ f(0) = 0,$$

where $H(t-\tau)$ is the unit-step function delayed by $\tau > 0$ units of time.
Solution: Taking the Laplace transform of each term with nonzero initial conditions and solving for $F(s)$, we have

$$F(s) = \frac{1}{(s-1)^3} + \frac{e^{-s\tau}}{s(s-1)^3}.$$

Using the method of partial fraction expansion, we can write

$$F(s) = \frac{1}{(s-1)^3} - \frac{e^{-s\tau}}{(s-1)^3} + \frac{e^{-s\tau}}{(s-1)^2} - \frac{e^{-s\tau}}{s-1} + \frac{e^{-s\tau}}{s}.$$

Taking the inverse Laplace transform of each term, we find

$$f(t) = \frac{1}{2}t^2 e^t H(t) - \frac{1}{2}(t-\tau)^2 e^{t-\tau} H(t-\tau) - (t-\tau)e^{t-\tau} H(t-\tau) - e^{t-\tau} H(t-\tau) + H(t-\tau).$$

Q-3) Determine the image in the w plane of the triangular region bounded by the lines $y = \pm x$ and $x = 1$ in the z-plane under the mapping $w = z^2$, by (i) indicating the corresponding image of each side and of each vertex of the triangle and (ii) showing clearly that each point in your image is indeed an image.

Solution: The vertices are at $A = (0,0)$, $B = (1,1)$, and $C = (1,-1)$ and are mapped by

$$u = x^2 - y^2, \quad v = 2xy$$

onto $A' = (0,0)$, $B' = (0,2)$, and $C' = (0,-2)$, respectively. The sides on $y = x$ and $y = -x$ are clearly mapped onto $u = 0,\ 0 \le v \le 2$ and $u = 0,\ -2 \le v \le 0$, respectively. The third side $x = 1,\ -1 \le v \le 1$ is mapped onto the part of the parabola

$$u = 1 - \frac{v^2}{4} \tag{1}$$

in the right half w-plane. A line $L:\ x = x_0,\ -x_0 \le y \le x_0$ in the triangular region has the image $L':\ u = x_0^2 - \frac{v^2}{4x_0^2}$ so that the region between the line $u = 0,\ -2 \le v \le 2$ and the parabola (**??**) is filled in a one-to-one manner as x_0 varies between 0 and 1.

Q-4) Find the temperature $T(x, y)$ in the thin metal plate between two circles

$$C_1 : (x - \frac{1}{4})^2 + y^2 = \frac{1}{16}, \quad C_2 : (x - \frac{1}{2})^2 + y^2 = \frac{1}{4},$$

if C_1 is maintained at one hundred degree Celcius and C_2 is maintained at zero degree Celcius.
HINT: Use the linear fractional transformation which maps $x = 0, 1/2, 1$ on the real line in z-plane onto $u = \infty, 1, 0$ on the real line in w-plane, respectively.
Solution: The transformation $w = \frac{1-z}{z}$ is the requires LFT and is such that

$$u = \frac{x - x^2 - y^2}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

Thus, it maps C_1 ($y^2 = -x^2 + 0.5x^2$) onto the line $u = 1$, $-\infty < v < \infty$ and C_2 ($y^2 = -x^2 + x$) onto the line $u = 0$, $-\infty < v < \infty$. Also the region between C_1 and C_2 is mapped onto the infinite strip between the lines. The temperature function $T(u, v) = 100u$ is bounded harmonic in this strip and satisfies the boundary conditions on the vertical lines. Therefore, the required temperature function in the z-plane is

$$T(x, y) = 100\frac{x - x^2 - y^2}{x^2 + y^2}.$$