Date: April 19, 2008, Saturday NAME:...
Time: 10:00-12:00
Özgüler STUDENT NO:...

Math 206 Complex Calculus – Midterm Exam II

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) (i) [20 pts.] Determine the Laurent series expansion for

$$f(z) = \frac{cosh(z)}{z(z^2-1)}$$

about $z_0 = 0$ by writing the coefficients $a_n,\ n \geq 0,\ b_n,\ n \geq 1$ and indicate the region in which the expansion is valid.
(ii) [5 pts.] Find the residue at the origin of $f(z)$.

Solution: (i) Writing the McLaurin expansions for $cosh(z)$ and $1/(1-z^2)$, we get

$$f(z) = -\frac{1}{z}(\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!})(\sum_{n=0}^{\infty}z^{2n}) = -\frac{1}{z}[\sum_{n=0}^{\infty}(\sum_{k=0}^{n}\frac{1}{(2k)!})z^{2n}] = -\frac{1}{z} - \sum_{n=0}^{\infty}(\sum_{k=0}^{n}\frac{1}{(2k)!})z^{2n+1}.$$

Thus

$$a_n = -\sum_{k=0}^{n}\frac{1}{(2k)!},\ n = 0, 1, 2, ...;\ b_1 = -0.5,\ b_n = 0,\ n = 2, 3, 4, ...$$

The expansion is valid for $0 < |z| < 1$.

(ii) The residue is the coefficient b_1 of $1/z$, i.e., -0.5.

Q-2) (i) [15 pts.] Determine and classify (pole?, removable?, essential?, not-isolated?) all singularities of

$$f(z) = \frac{e^z}{sinh(\pi/z)}.$$

(ii) [10 pts.] Determine the residue at each isolated singularity of this $f(z)$.

Solution: (i) Since e^z is entire, the singularities are at $z_0 = 0$ and at $z_n = 1/(in)$, $n = \pm 1, \pm 2, \ldots.$ All are isolated singularities and simple poles of $f(z)$ *except* $z_0 = 0$ which is not isolated.

(ii) The residues at z_n, $n \neq 0$ are given by $p(z) := e^z$, $q(z) := sinh(\pi/z)$, and for $n = \pm 1, \pm 2, \ldots$:

$$Res_{z_n}[f(z)] = \frac{p(z_n)}{q'(z_n)} = -\frac{z_n^2}{\pi}\frac{e^{z_n}}{cosh(\pi/z_n)} = \frac{1}{n^2\pi}\frac{cos(1/n) - isin(1/n)}{(-1)^n}$$

$$= \frac{(-1)^n}{n^2\pi}[cos(1/n) - isin(1/n)].$$

Q-3) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x\ sin(3x)}{x^4+16}\ dx$$

using complex integration on a suitable contour. *Important note: Computation of any limits you may use in your derivation should all be explicitly shown.*

Solution: Let $f(z) := \frac{ze^{i3z}}{z^4+16}$ and consider the simple closed positively oriented contour $C_1 + C_R$ with

$$C_1 : z = x,\ -R \leq x \leq R;\ \ C_R : z = Re^{i\theta},\ 0 \leq \theta \leq \pi.$$

For $R > 2$, the contour contains the two simple poles

$$c_0 = 2e^{i\pi/4}, c_1 = 2e^{i3\pi/4}$$

of $f(z)$ and no other singularities. On C_r,

$$|\frac{z}{z^4+16}| \leq \frac{R}{R^4-16},$$

which has limit 0 as $R \to \infty$. By Jordan's Lemma, the integral on C_R of $f(z)$ tends to zero as $R \to \infty$. On C_1, the integral gives the improper integral below as $R \to \infty$ so that, by the Residue Theorem, we have

$$\int_{-\infty}^{\infty} \frac{x\ e^{i3x}}{x^4+16}\ dx = 2i\pi(Res_{c_0}[f(z)] + Res_{c_1}[f(z)])$$

$$= 2i\pi(\frac{e^{i3\sqrt{2}}e^{-3\sqrt{2}}}{16i} + \frac{e^{-i3\sqrt{2}}e^{-3\sqrt{2}}}{-16i})$$

$$= i\frac{\pi}{4}\ e^{-3\sqrt{2}}\ sin(3\sqrt{2}).$$

Taking the imaginary part of both sides, we get

$$\int_{-\infty}^{\infty} \frac{x\ sin(3x)}{x^4+16}\ dx = \frac{\pi}{4}\ e^{-3\sqrt{2}}\ sin(3\sqrt{2}).$$

Q-4) Evaluate the integral

$$\int_0^\infty \frac{(ln\, x)^2}{x^2+1}\, dx$$

using complex integration on a suitable contour. *Important note: Computation of any limits you may use in your derivation should all be explicitly shown.*

HINT: You can use the fact that

$$\int_0^\infty \frac{1}{r^2+1}\, dr = \frac{\pi}{2}.$$

Solution: Let $f(z) = \frac{log\, z)^2}{z^2+1}$, where $log\, z$ is such that $-\pi/2 < arg\, z < 3\pi/2$, and consider the simple positively oriented closed contour $L_1 + C_\rho + L_2 + C_R$ with parametrizations

$$-L_1 : z = re^{i\pi},\ \rho \le r \le R;\ \ C_R : z = Re^{i\theta},\ 0 \le \theta \le \pi;$$

$$L_2 : z = r,\ \rho \le r \le R;\ -C_\rho : z = \rho e^{i\theta},\ 0 \le \theta \le \pi.$$

The function $f(z)$ has the pole i inside the contour for $\rho < 1,\ R > 1$. It is easy to see that

$$\int_{L_1} f(z)\, dz = \int_\rho^R \frac{(ln\, r + i\pi)^2}{r^2+1}\, dr, \quad \int_{L_2} f(z)\, dz = \int_\rho^R \frac{(ln\, r)^2}{r^2+1}\, dr.$$

Also

$$|\int_{C_\rho} f(z)\, dz| = |\int_0^\pi \frac{(ln\, \rho + i\theta)^2}{\rho^2 e^{i2\theta}+1} i\rho e^{i\theta}\, d\theta| \le \frac{(\pi - ln\, \rho)^2}{1-\rho^2}\pi\rho,$$

for $\rho < 1$, where the right hand side goes to zero as $\rho \to 0$ (as can be shown using L'Hopital's rule) and

$$|\int_{C_R} f(z)\, dz| = |\int_0^\pi \frac{(ln\, R + i\theta)^2}{R^2 e^{i2\theta}+1} iRe^{i\theta}\, d\theta| \le \frac{(\pi + ln\, R)^2}{R^2-1}\pi R,$$

for $R > 1$, which goes to zero as $R \to \infty$, again by L'Hopital's rule. Putting these together and taking limits as $\rho \to 0$ and as $R \to \infty$, we obtain

$$2\int_0^\infty \frac{(ln\, r)^2}{r^2+1}\, dr + \int_0^\infty \frac{-\pi^2}{r^2+1}\, dr + \int_0^\infty \frac{2i\pi\, ln\, r}{r^2+1}\, dr = 2i\pi\ Res_{z=i}[f(z)] = -\frac{\pi^3}{4}.$$

Using the HINT and taking the real part of both sides, we finally get

$$\int_0^\infty \frac{(ln\, r)^2}{r^2+1}\, dr = \frac{\pi^3}{8}.$$