Date: January, [illegible], Saturday NAME:...

Time: 15:30-17:30

STUDENT NO:...

Math 206 Complex Calculus – Final Exam

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:

Check that there are 4 questions on your exam booklet.

No correct answer without a satisfying reasoning is accepted. Show your work in detail.

Write your name on the top of every page.

Q-1) Consider the difference equation

$$y(n+2) - 4y(n+1) + 4y(n) = 2^n,\ y(0) = 1,\ y(1) = -1.$$

(i) (5 pts.) Write down the values of $y(2)$, $y(3)$, $y(4)$.

(ii) (20 pts.) Determine the solution $y(n)$ of the equation.

Solution:

(i)

$$y(2) = 4y(1) - 4y(0) + 2^0 = -7,$$
$$y(3) = 4y(2) - 4y(1) + 2 = -22,$$
$$y(4) = 4y(3) - 4y(2) + 2^2 = -56.$$

(ii) Taking the Z-Transform of every term, we have

$$(z^2 - 4z + 4)Y(z) = \frac{z}{z-2} + z^2 - 5z,$$

where the last two terms are due to nonzero initial conditions. Thus,

$$Y(z) = \frac{z(z^2 - 7z + 11)}{(z-2)^3}.$$

Let us use the method of residues to find the inverse transform:

$$y(n) = \mathrm{Res}_{z=2}[z^{n-1}Y(z)] = \frac{1}{2}\frac{d^2}{dz^2}[z^n 9z^2 - 7z + 11)]|_{z=2} = 2^{n-3}(n^2 - 13n + 8).$$

Q-2) Solve the linear system of differential equations

$$2\frac{dx}{dt} + \frac{dy}{dt} - x - y = e^t,$$
$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = e^{-t}, \quad x(0) = 2,\ y(0) = 1$$

Solution: Taking the Laplace transform of every term and substituting the initial values, we get

$$(2s-1)X(s) + (s-1)Y(s) = \frac{1}{s-1} + 5$$

$$(s+2)X(s) + (s+1)Y(s) = \frac{1}{s+1} + 3,$$

which, when solved for $X(s), Y(s)$ give

$$X(s) = \frac{4s}{(s^2-1)(s^2+1)} + \frac{2(s+4)}{s^2+1},\ Y(s) = \frac{s^2-6s-1}{(s^2-1)(s^2+1)} + \frac{s-13}{s^2+1}.$$

Since

$$\frac{4s}{(s^2-1)(s^2+1)} = \frac{1}{s-1} + \frac{1}{s+1} - \frac{2s}{s^2+1},$$

we can write

$$X(s) = \frac{1}{s-1} + \frac{1}{s+1} - \frac{2s}{s^2+1} + \frac{2(s+4)}{s^2+1} = \frac{1}{s-1} + \frac{1}{s+1} + \frac{8}{s^2+1}$$

and

$$Y(s) = \frac{1}{s^2+1} - \frac{6s}{(s^2-1)(s^2+1)} + \frac{s-13}{s^2+1} = \frac{s-12}{s^2+1} - \frac{3}{2}\left(\frac{1}{s-1} + \frac{1}{s+1} - \frac{2s}{s^2+1}\right).$$

Taking the inverse transforms, we arrive at

$$x(t) = e^t + e^{-t} + 8sin(t),\ y(t) = -\frac{3}{2}(e^t + e^{-t}) + 4cos(t) - 12sin(t).$$

Q-3) Find all possible solutions to the differential equation

$$\frac{d^4x}{dt^4} + 8\frac{d^2x}{dt^2} + 16x(t) = \delta(t),$$

where $\delta(t)$ is the Dirac delta function and it is given that

$$x^{(3)}(0) \text{ is arbitrary, } x^{(2)}(0) = 0,\ x^{(1)}(0) = 0,\ x(0) = 0.$$

Solution: Taking the Laplace transform of each term, we have

$$(s^4 + 8s^2 + 16)X(s) = 1 + x^{(3)}(0),\ \ X(s) = \frac{k}{(s^2+4)^2},$$

where $k := 1 + x^{(3)}(0)$. Let us use the method of residues to find the inverse Laplace transform:

$$x(t) = \mathrm{Res}_{z=2i}[e^{st}X(s)] + \mathrm{Res}_{z=-2i}[e^{st}X(s)] = r + \bar{r},$$

where $r := \mathrm{Res}_{z=2i}[e^{st}X(s)]$. Now,

$$r = [\frac{d}{ds}(\frac{e^{st}}{(s+2i)^2})]|_{s=2i} = \frac{te^{st}(s+2i) - 2e^{st}}{(s+2i)^3}|_{s=2i} = \frac{4ite^{2it} - 2e^{2it}}{(4i)^3},$$

so that

$$x(t) = \frac{4ite^{2it} - 2e^{2it}}{(4i)^3} + \frac{-4ite^{-2it} - 2e^{-2it}}{(-4i)^3} = \frac{1}{16}sin(2t) - \frac{1}{8}t\,cos(2t).$$

Q-4) Determine the value of the improper integral

$$\int_0^\infty \frac{sin(2x)\,dx}{x\,(x^2+9)},$$

by contour integration for a suitably chosen simple closed contour in z-plane. If you evaluate certain limits in your derivation, then show all steps of your evaluation clearly.

Solution: Let us consider

$$f(z) = \frac{exp(2iz)}{z(z^2+9)},$$

and the contour $C = L_1 + C_R + L_2 + C_\rho$, covering the negative and positive x-axes by L_2, L_1 and consisting of the semicircles C_ρ and C_R with ρ small and R large. Then, parametrizing the integrals along L_1 and L_2, and using the residue theorem, we can write

$$\int_C f(z)\,dz = \int_\rho^R \frac{e^{i2r}}{r(r^2+9)}\,dr - \int_\rho^R \frac{e^{-i2r}}{r(r^2+9)}\,dr + \int_{C_R} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 2i\pi\ Res_{z=3i} f(z). \tag{1}$$

We have

$$2i\pi\ Res_{z=3i} f(z) = -2i\pi\,\frac{exp(-6)}{18}.$$

Also, the integral on C_R vanishes as $R \to \infty$ by Jordan's Lemma, since on C_R, $z = Rexp(i\theta)$ and $|f(z)exp(-2iz)| \le 1/R(R^2-9)$, which has limit zero as $R \to \infty$. The integral on C_ρ is given by $-i\pi\,B_0$, where B_0 is the residue of $f(z)$ at its simple real pole at the origin. Now, $B_0 = Res_{z=0} f(z) = 1/9$ so that

$$\int_{C_\rho} f(z)\,dz = \frac{-i\pi}{9}.$$

Therefore, taking limits as $\rho \to 0,\ R \to \infty$ in (1), we get

$$2\,i \int_0^\infty \frac{sin(2r)}{r(r^2+9)}\,dr = \frac{i\pi}{9} + 2i\pi\frac{e^{-6}}{18}$$

which gives

$$\int_0^\infty \frac{sin(2r)}{r(r^2+9)}\,dr = \frac{1-e^{-6}}{18}.$$

Date: May 24, 2005; Tuesday NAME:..
Instructors: Sertöz and Özgüler
Time: 16.00-18.00 STUDENT NO:..

Math 206 Complex Calculus–Final Exam

1	2	3	4	5	TOTAL
20	20	20	20	20	100

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PLEASE READ:
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Q-1) Determine the inverse Z-transform of

$$F(z) = \frac{e^{1/z}}{z-2}.$$

Solution: Use the method of residues: $f(n) = \sum Res[z^{n-1}F(z)]$. Two singularitites of $z^{n-1}F(z)$ are at $z=0$ and at $z=2$, both simple. Now, $Res_{z=0}[z^{n-1}F(z)]$ is the coefficient of z^{-1} in the product of series

$$z^{-1}\frac{1}{1-(2/z)} = z^{-1} + 2z^{-2} + 2^2z^{-3} + ... + 2^{n-1}z^{-n} + 2^nz^{-(n+1)} + 2^{n+1}z^{-(n+2)} + ...$$

and

$$z^{n-1}e^{1/z} = z^{n-1} + z^{n-2} + \frac{1}{2!}z^{n-3} + ... + \frac{1}{(n-1)!} + \frac{1}{n!}z^{-1} + \frac{1}{(n+1)!} + z^{-2} + ...$$

which is

$$\sum_{k=0}^{k=n-1} \frac{2^k}{(n-k-1)!}.$$

On the other hand, $Res_{z=2}[z^{n-1}F(z)] = 2^{n-1}e^{0.5}$ so that

$$f(n) = \sqrt{e}2^{n-1} + \sum_{k=0}^{k=n-1} \frac{2^k}{(n-k-1)!}.$$

NAME: STUDENT NO:

Q-2) Consider the sequence $1, 1, 2, 4, 7, 11, 16, ...$ that begins with $n = 0$ and satisfies $f(n+1) - f(n) = n$. Find $f(n)$.

Solution: $zF(z) - z + F(z) = z/(z-1)^2$ gives $F(z) = z/(z-1)^3 + z/(z-1)$. Now, $Z^{-1}\{z/(z-1)^3\} = Res_{z=1}[z^n/(z-1)^3] = n(n-1)/2$. Also, $Z^{-1}\{z/(z-1)\} = 1$. Hence,

$$f(n) = \frac{n(n-1)}{2} + 1.$$

NAME: STUDENT NO:

Q-3) Find a conformal map which maps the interior of the set $\{x + iy \in \mathbb{C} | y \geq 0, 0 \leq x \leq \pi/2\}$ onto the interior of the unit disk such that the point $\pi/4 + i$ is mapped to the origin.

Solution: $w_1 = \sin z$ maps the region onto the first quadrant. $w_2 = w_1^2$ maps the first quadrant onto the upper half plane. $w = (w_2 - z_0)/(w_2 - \overline{z_0})$ maps the first quadrant onto the unit circle such that z_0 is mapped to the origin. We don't need the $\exp(i\alpha)$ factor. Now follow what happens to the given point to find precisely what z_0 should be. It turns out that $z_0 = \frac{1}{2}(1 + i \sinh 2)$.

NAME: STUDENT NO:

Q-4) Using a complex logarithmic mapping,
a) find a bounded harmonic function $H(x, y)$, or $H(r, \theta)$, in the wedge $0 < arg(z) < \pi/6$, $|z| > 0$ such that $H(r, 0) = 0$ and $H(r, \pi/6) = 1$ for $r > 0$.
b) Find a harmonic conjugate $G(x, y)$ of $H(x, y)$ and describe the families of level curves $H(x, y) = c_1$, $G(x, y) = c_2$ for real constants c_1, c_2.

Solution: a) $H(u, v) = Re\{-i\frac{6}{\pi}Log\, z\}$ in w-plane so that $H(x, y) = \frac{6}{\pi}arctan(y/x)$ with the range of arctan function taken between 0 and π.
b) $G(u, v) = Im\{-i\frac{6}{\pi}Log\, z\}$ in w-plane so that $G(x, y) = \frac{6}{\pi}ln(\sqrt{x^2 + y^2})$. Hence, $H(x, y) = c_1$ give radial lines and $G(x, y) = c_2$ give circular arcs in the wedge.

Date: December [illegible], Saturday NAME:...

Time: 12:15-14:15

STUDENT NO:...

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Q-1) Consider the difference equation

$$y(n+2) + a^2\, y(n) = \delta(n), \;\; y(0) = 0, \;\; y(1) = 1$$

where a is a real number and $\delta(n)$ denotes the function $\delta(0) = 1$ and $\delta(n) = 0, \; n \neq 0$.

(i) (5 pts.) Write down the values of $y(2)$, $y(3)$, $y(4)$.

(ii) (20 pts.) Determine the solution $y(n)$ of the difference equation.

Solution: (i) $y(2) = 1, \; y(3) = -a^2, \; y(4) = -a^2$.

(ii) If $Y(z)$ is the Z-transform of $y(n)$, then as $Z[\delta(z)] = 1$, we have

$$(z^2 + a^2)Y(z) = 1 + z, \;\; Y(z) = \frac{z+1}{z^2+a^2}.$$

Let us use Partial Fraction Expansion to find the inverse Z-transform of $Y(z)$. Then,

$$\frac{Y(z)}{z} = \frac{a^{-2}}{z} + \frac{ia-1}{2a^2(z+ia)} + \frac{ia+1}{2a^2(z-ia)},$$

or

$$Y(z) = a^{-2} + \frac{(ia-1)z}{2a^2(z+ia)} + \frac{(ia+1)z}{2a^2(z-ia)} \;\Rightarrow\; y(n) = \frac{1}{a^2}\delta(n) + \frac{ia-1}{2a^2}\,(-ia)^n + \frac{ia+1}{2a^2}\,(ia)^n.$$

Considering even and odd n:

$$y(n) = \begin{cases} \frac{1}{a^2}\delta(n) + a^{n-2}(-1)^{(n-2)/2}, & n \text{ even} \\ \frac{1}{a^2}\delta(n) + a^{n-1}(-1)^{(n-1)/2}, & n \text{ odd.} \end{cases}$$

Q-2) Given that the Laplace Transform of $f(t)$ is

$$F(s) = \frac{4}{(s^2+9)^2},$$

find $f(t), t \geq 0$ using any method you like.

Solution: Let us use the method of residues. Since $|F(s)| \leq \frac{4}{(R^2-9)^2}$ for all $s = R\,exp(i\theta)$, $\lim_{R\to\infty} |F(s)| = 0$ on the contour $C_{R,\gamma}$ (see the book) and we can use the Residue method. According to this method, $f(t) = Res_{s=i3}[exp(st)\,F(s)] + Res_{s=-i3}[exp(st)\,F(s)]$. Since,

$$Res_{s=i3}[exp(st)\,F(s)] = \frac{d}{ds}\,|\frac{e^{st}}{(s+i3)^2}|_{s=i3}, \quad Res_{s=-i3}[exp(st)\,F(s)] = \frac{d}{ds}\,|\frac{e^{st}}{(s-i3)^2}|_{s=-i3},$$

we have

$$Res_{s=i3}[exp(st)\,F(s)] = -\frac{4}{4\,3^2}te^{i3t} - i\frac{4}{4\,3^3}e^{i3t},$$

and residue at $s = -i3$ is the conjugate of this. Therefore,

$$f(t) = -\frac{1}{9}te^{i3t} - i\frac{1}{27}e^{i3t} - \frac{1}{9}te^{-i3t} + i\frac{1}{27}e^{-i3t} = -\frac{2}{9}tcos(3t) + \frac{2}{27}sin(3t).$$

Q-3) Determine the value of the improper integral

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)},$$

by contour integration for a suitably chosen simple closed contour in z-plane. If you evaluate certain limits in your derivation, show all steps of your evaluation clearly.

Solution: Let I denote the integral we are asked to find. Let us consider

$$f(z) = \frac{exp(-0.5\,log z)}{z^2+1}, \quad |z|>0, \; -\pi/2 < arg\, z < 3\pi/2$$

and the contour $C = L_1 + C_R + L_2 + C_\rho$, covering the negative and positive x-axes by L_2, L_1 and consisting of the semicircles C_ρ and C_R with ρ small and R large. Then,

$$\int_C f(z)\,dz = \int_\rho^R f(r)\,dr + \int_R^\rho e^{-i\pi/2} f(r)\,dr + \int_{C_R} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 2i\pi \; Res_{z=i} f(z). \quad (1)$$

We have

$$2i\pi \; Res_{z=i} f(z) = 2i\pi \, \frac{exp(-0.5\,log\, i)}{i+i} = \pi\, exp(-i\pi/4).$$

Also, the integral on C_R vanishes as $R\to\infty$, since on C_R, $z = Rexp(i\theta)$ and $|f(z)| \le R^{-0.5}/(R^2-1)$, which has limit zero as $R\to\infty$. Similarly, the integral on C_ρ vanishes as $\rho \to 0$, since, by $z = \rho\, exp(i\pi - i\theta)$,

$$|\int_{C_\rho} f(z)\,dz| \le \int_0^\pi \frac{\rho^{-0.5}\rho}{1-\rho^2} d\theta \le \frac{\pi\rho^{0.5}}{1-\rho^2}$$

which has limit zero as $\rho\to 0$. Therefore, taking limits as $\rho\to 0,\; R\to\infty$ in (1), we get

$$I + e^{i\pi/2}\, I = \pi\, exp(-i\pi/4) \;\Rightarrow\; I = \frac{\pi\, exp(-i\pi/4)}{1+e^{-i\pi/2}} = \frac{\pi}{2cos(\pi/4)} = \frac{\pi}{\sqrt{2}}.$$

Q-4) (i) (10 pts.) Find a linear transformation that maps the region to the left of the line $x+y=1$ in z-plane onto the lower half of the Z-plane, such that the point $z=1$ is mapped onto $Z=0$ and the point $z=i$ is mapped onto $Z=-\sqrt{2}$.

(ii) (10 pts.) Find a linear fractional transformation that maps the lower half of the Z-plane onto the unit disk in w-plane in such a way that the images of points $0, -\sqrt{2}, -exp(i\pi/4)$ in Z-plane are, respectively, the points $1, exp(i\pi/4), 0$ in w-plane.

(iii) (5pts.) Determine the composition of the two transformations you found in (i) and (ii).

Solution: (i) It is clear that we need a rotation by $\pi/4$ and a suitable translation. Hence, $Z = T_1(z) = exp(i\pi/4)(z-1)$ is the required transformation, since $T_1(1) = 0$ and $T_1(i) = exp(i\pi/4)(i-1) = -\sqrt{2}$. Also note that, the origin, e.g., is mapped onto $-exp(i\pi/4)$, which is in the lower half Z-plane.

(ii) There is a unique LFT which maps three points to three points. Assuming $c \neq 0$ and starting with

$$w = T_2(Z) = \frac{aZ+b}{Z+d},$$

substitutions $T_2(0) = 1,\ T_2(-exp(i\pi/4)) = 0$ give

$$T_2(Z) = \frac{a[Z + exp(i\pi/4)]}{Z + a\,exp(i\pi/4)}.$$

Substituting $T_2(-\sqrt{2}) = exp(i\pi/4)$, we obtain

$$a = \frac{\sqrt{2}}{exp(i\pi/4) - i}.$$

(iii) $w = T(z) = T_2(T_1(z))$ is given by

$$w = T(z) = \frac{\sqrt{2}\,z}{[exp(i\pi/4) - i](z-1) + \sqrt{2}}.$$

MATH 206; FINAL EXAM; January 3, 2005

SOLUTIONS

Q.1 Let $z = e^{i\theta}$ so that $\cos\theta = \frac{z+z^{-1}}{2}$, $d\theta = \frac{dz}{iz}$:

$$\int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\, d\theta = \int_{C} \frac{z+z^{-1}}{2\left(2+\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} = \frac{1}{i}\int_C \frac{z^2+1}{z(z^2+4z+1)}$$

(C: unit circle)

The roots of z^2+4z+1 are $-2\mp\sqrt{3}$ of which $\sqrt{3}-2$ is inside the unit-circle C. Hence,

$$\int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\, d\theta = 2\pi\left[\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=\sqrt{3}-2} f(z)\right], \quad f(z) := \frac{z^2+1}{z(z^2+4z+1)}.$$

We have,

$$\operatorname{Res}_{z=0} f(z) = \left.\frac{z^2+1}{z^2+4z+1}\right|_{z=0} = 1, \quad \operatorname{Res}_{z=\sqrt{3}-2} \left.\frac{z^2+1}{z(z+2+\sqrt{3})}\right|_{z=\sqrt{3}-2} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}$$

Therefore,

$$\boxed{\int_0^{2\pi} \frac{\cos\theta}{(2+\cos\theta)}\, d\theta = 2(3+\sqrt{3})\pi}$$

Q.2. Let $x_0 := x(0)$, $y_0 = y(0)$ and take Laplace Transform:

$$sX(s) - x_0 = -2X(s) - Y(s), \quad sY(s) - y_0 = X(s)$$

$$\Rightarrow \left.\begin{aligned}(s+2)X(s) &= -Y(s) + x_0 \\ sY(s) &= X(s) + y_0\end{aligned}\right\} \quad \begin{aligned} X(s) &= \frac{s}{(s+1)^2}x_0 - \frac{1}{(s+1)^2}y_0 \\ Y(s) &= \frac{s+2}{(s+1)^2}y_0 + \frac{1}{(s+1)^2}x_0\end{aligned}$$

$$\Rightarrow X(s) = \frac{x_0}{s+1} - \frac{1}{(s+1)^2}(x_0+y_0), \quad Y(s) = \frac{y_0}{s+1} + \frac{1}{(s+1)^2}(x_0+y_0)$$

Taking inverse L.T.: All solutions are

$$x(t) = e^{-t}x_0 - te^{-t}(x_0+y_0)$$
$$y(t) = e^{-t}y_0 + te^{-t}(x_0+y_0), \quad t \geq 0$$

and x_0, y_0 are arbitrary constants.

Q.3. Taking z-transform

$$z^3 X(z) - z^3 x(0) - z^2 x(1) - z X(z) - 2z^2 X(z) + 2z^2 x(0) + 2z x(1) - z X(z) + 2z X(0) + 2 x(2) = \frac{z}{z-1}$$

$$\Rightarrow (z^3 - 2z^2 - z + 2) \cdot X(z) = \frac{z}{z-1} + z^3 - z^2$$

$$\Rightarrow X(z) = \frac{z}{(z-1)(z^2-1)(z-2} + \frac{z^2}{(z+1)(z-2)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{z(z-1)^2+1}{(z-1)^2 (z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)}$$

$A = \frac{1}{4}$, $B = 1$, $C = -\frac{1}{2}$

$$D = \frac{\left[(z-1)^2 + 2z(z-1)\right] \cdot \left[(z-1)^2 (z+1)(z-2)\right] - \left[(2z-1) \cdot (z(z-1)^2+1)\right]}{\left[(z+1)(z-2)\right]^2} \Bigg|_{z=1} = -\frac{1}{4}$$

$$X(z) = \frac{1}{4} \cdot \frac{z}{z+1} + \frac{z}{z-2} - \frac{1}{2} \cdot \frac{z}{(z-1)^2} - \frac{1}{4} \cdot \frac{z}{z-1}$$

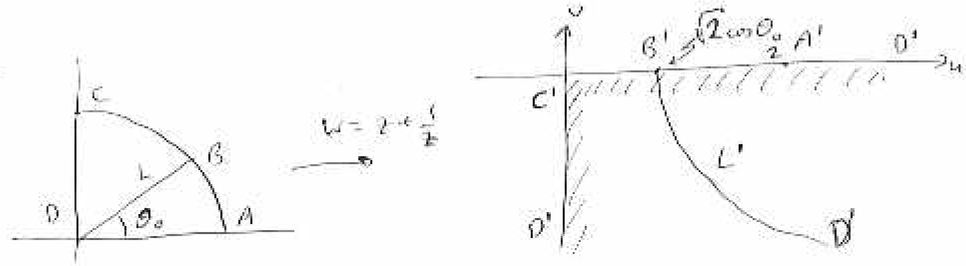
$$\Rightarrow \boxed{x(n) = \frac{1}{4} (-1)^n + 2^n - \frac{1}{2} n - \frac{1}{4}, \quad n \geq 0}$$

Alternatively,

$$x(n) = \sum_{n=1}^{3} \operatorname{Res}_{z=z_i} \left[\frac{z^n \left[z(z-1)^2+1\right]}{(z-1)^2 (z+1)(z-2)}\right], \quad z_1 = 1,\ z_2 = -1,\ z_3 = 2$$

$$= \frac{-2n-1}{4} + (2)^n + \frac{1}{4}(-1)^n$$

Q.4.

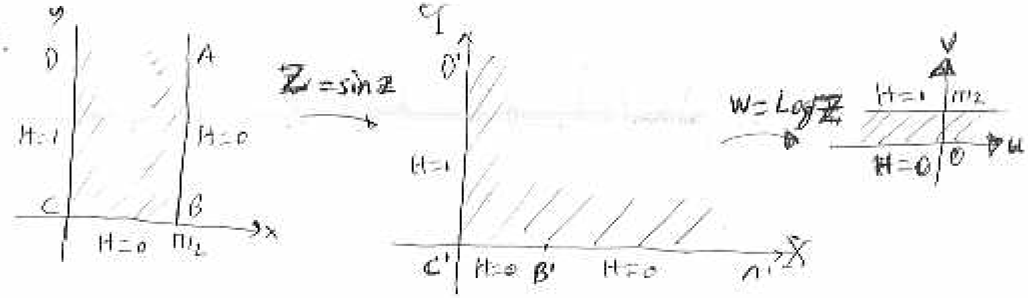


Let $z=re^{i\theta}$. Then, $u=(r+\frac{1}{r})\cos\theta$, $v=(r-\frac{1}{r})\sin\theta$.

L: $z=re^{i\theta_0}$ $(0<r<1)$ $\Rightarrow$ $\frac{u^2}{\cos^2\theta_0}-\frac{v^2}{\sin^2\theta_0}=(r+\frac{1}{r})^2-(r-\frac{1}{r})^2=2$

Hence, image of L is part of the hyperbola of foci $\sqrt{2}$ in the fourth quadrant (since $u>0$, $v<0$ for $r<1$). As L varies from $\theta_0=0$ to $\theta_0=\pi/2$, the whole of the fourth quadrant is covered.

Q.5.



Hence, $H(u,v)=\frac{2}{\pi}v=\frac{2}{\pi}\arg Z=\frac{2}{\pi}\tan^{-1}\frac{Y}{X}$ $(0\le\tan^{-1}\le\pi/2)$.

But $Y=\cos x\sinh y$, $X=\sin x\cosh y$, so that

$$H(x,y)=\frac{2}{\pi}\tan^{-1}\left(\frac{\tanh y}{\tan x}\right),\quad 0\le\tan^{-1}\le\pi/2.$$

Note that, $0\le H(x,y)\le 1$ as $0\le\tan^{-1}\le\pi/2$.

Date: 24 May 2004, Monday
Instructors: Özgüler & Sertöz
Time: 12:15-14:15

NAME:..

STUDENT NO:..

Your Section/Instructor:..

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Q-1) Find $\mathrm{Res}_{z=0}\dfrac{z}{z^2-\sinh z^2}$.

$$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$$

$$z^2 - \sinh z^2 = -\frac{z^6}{3!} - \frac{z^{10}}{5!} - \cdots = -\frac{z^6}{3!}\left(1 + \frac{3!}{5!}z^4 + \cdots\right)$$

$$\frac{z}{z^2-\sinh z^2} = \frac{1}{-\frac{z^5}{3!}\left(1+\frac{3!}{5!}z^4+\cdots\right)} = -\frac{3!}{z^5}\left(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 \cdots\right)$$

Residue is then $-3!\cdot b_4$.

Here $(b_0 + b_1 z + \cdots)\left(1 + \frac{3!}{5!}z^4 + \cdots\right) = 1.$

we easily find that $b_0 = 1$, $b_1 = b_2 = b_3 = 0$, $b_4 = -\frac{1}{20}$.

Then residue is $\frac{3}{10}$.

NAME: STUDENT NO:

Q-2) Calculate the Cauchy principal value of the integral $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$.

Let

$$f(z) = \frac{z}{(z^2+1)(z^2+2z+2)}$$

which has singularities $i, -1+i$ in the positively oriented contour consisting of the real segment $[-R, R]$ and the semicircle C_R of radius R in the upper half of the complex plane, with $R > \sqrt{2}$. By the Residue Theorem

$$P.V. \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = \lim_{R\to\infty} \int_{-R}^{R} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$$

$$= Re[2\pi\, i(B_1 + B_2)] - Re(\lim_{R\to\infty} \int_{C_R} f(z)dz),$$

where B_1 and B_2 are the residues at i and $-1+i$ of $f(z)$, respectively. Now

$$B_1 = \frac{z}{(z+i)(z^2+2z+2)}|_{z=i} = \frac{i}{2i(1+2i)} = \frac{1-2i}{10},$$

$$B_2 = \frac{z}{(z^2+1)(z+1+i)}|_{z=-1+i} = \frac{-1+3i}{10}.$$

On C_R, $z = R\,exp(i\theta)$, $0 \le \theta \le \pi$ so that

$$|\int_{C_R} f(z)dz| \le \frac{\pi\, R^2}{(R^2-1)[(R-1)^2-1]}.$$

It follows that

$$Re(\lim_{R\to\infty} \int_{C_R} f(z)dz) = 0$$

and

$$P.V. \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = Re[2i\pi(\frac{1-2i}{10} + \frac{-1+3i}{10})] = -\frac{\pi}{5}.$$

NAME: STUDENT NO:

Q-3) Using Laplace transform techniques solve the initial value problem

$$f''(t) + 2f'(t) + f(t) = \sin t, \quad \text{whith } f(0) = 3, \; f'(0) = 1.$$

$\mathcal{L}(f(t)) = F(s)$

$\mathcal{L}(f'(t)) = sF(s) - f(0) = sF(s) - 3$

$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0) = s^2F(s) - 3s - 1$

$\mathcal{L}(\sin t) = \dfrac{1}{1+s^2}$.

Transforming both sides of the differential equation gives

$$(s^2+2s+1)F(s) - (3s+7) = \frac{1}{1+s^2},$$

Solving for $F(s)$:

$$F(s) = \frac{3s^3+7s^2+3s+8}{(1+s^2)(1+s)^2}$$

$$= \frac{9}{2} \cdot \frac{1}{(1+s)^2} - \frac{1}{2} \cdot \frac{s}{1+s^2} + \frac{7}{2} \cdot \frac{1}{1+s}.$$

Applying inverse of Laplace transform:

$$f(t) = \frac{9}{2} \cdot te^{-t} - \frac{1}{2} \cdot \cos t + \frac{7}{2} \cdot e^{-t}.$$

NAME: STUDENT NO:

Q-4) Consider the mapping $w = z + \frac{1}{z}$. Describe the images of the following sets under this mapping:

i- The x-axis.
ii- The y-axis.
iii- A ray with angle θ where $0 < \theta < \pi/2$.
iv- A ray with angle θ where $\pi/2 < \theta < \pi$.

Letting $z = re^{i\theta}$, we get $w = \frac{1}{2}(r+\frac{1}{r})\cos\theta + i\cdot\frac{1}{2}(r-\frac{1}{r})\sin\theta = u+iv$

i) The positive x-axis corresponds to $\theta = 0$. It maps onto $u \geq 1$.
The negative x-axis corresponds to $\theta = \pi$. It maps onto $u \leq -1$.

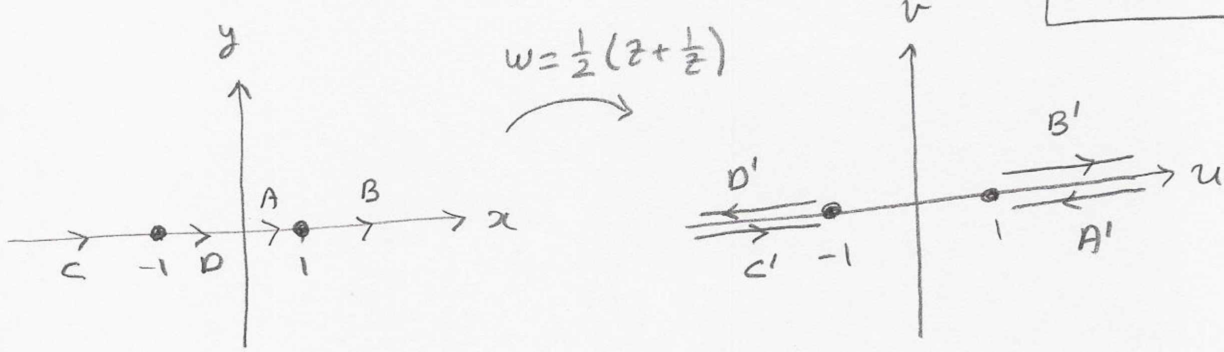
ii) The y-axis corresponds to $\theta = \pm\frac{\pi}{2}$. It maps onto v-axis.

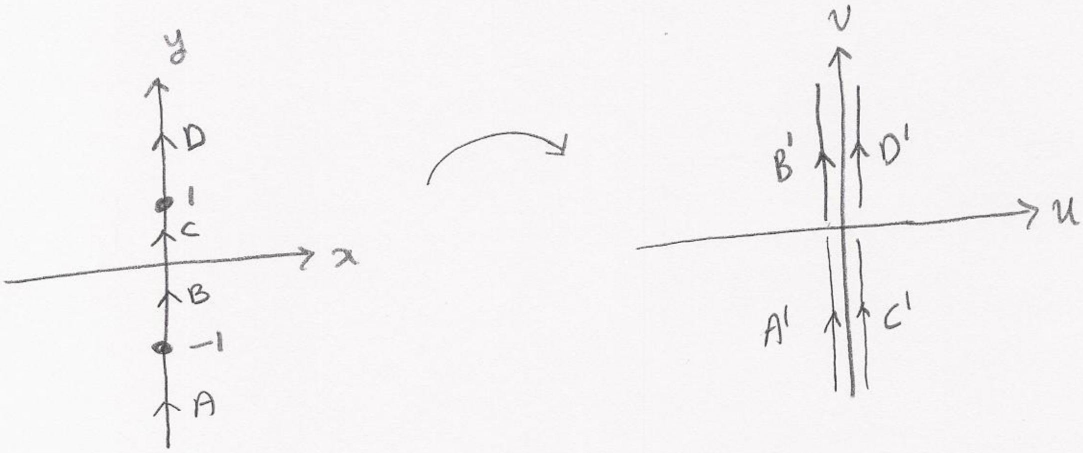
iii) A ray with angle θ maps onto one arm of the hyperpola $\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1$. Signs of $\cos\theta$ and $\sin\theta$ detemine which arm is the target and in which direction it is traversed as r goes from 0 to ∞.
when $0 < \theta < \pi/2$, both $\cos\theta$ and $\sin\theta$ are positive, and the right arm of the hyperbola is traversed.

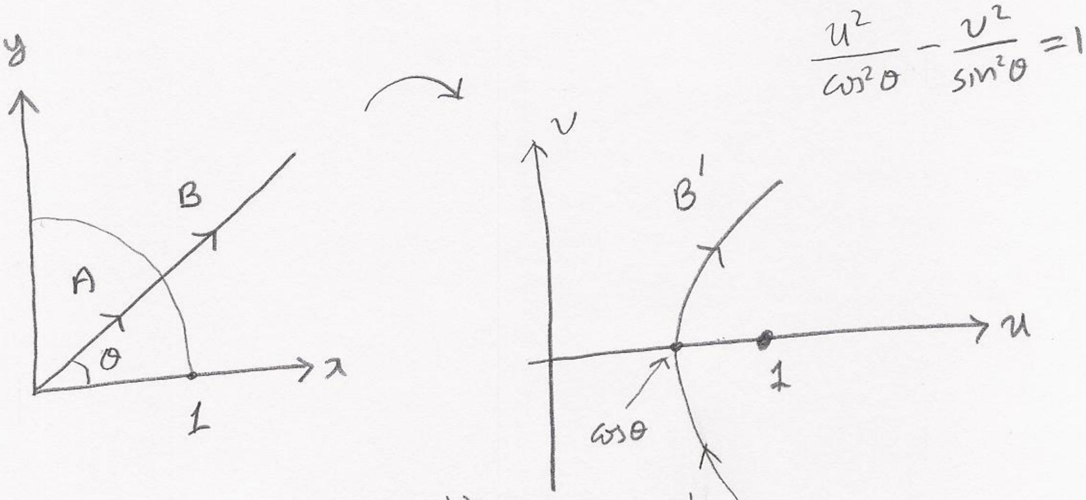
iv) since $\cos(\pi-\theta) = -\cos\theta$ and $\sin(\pi-\theta) = \sin\theta$, the other arm of the hyperbola in (iii) is traversed.

In both (iii) and (iv) keep in mind that θ is fixed,
$0 < r < \infty$ and $u = \frac{1}{2}(r+\frac{1}{r})\cos\theta$,
$v = \frac{1}{2}(r-\frac{1}{r})\sin\theta$.

Figures for Q-4

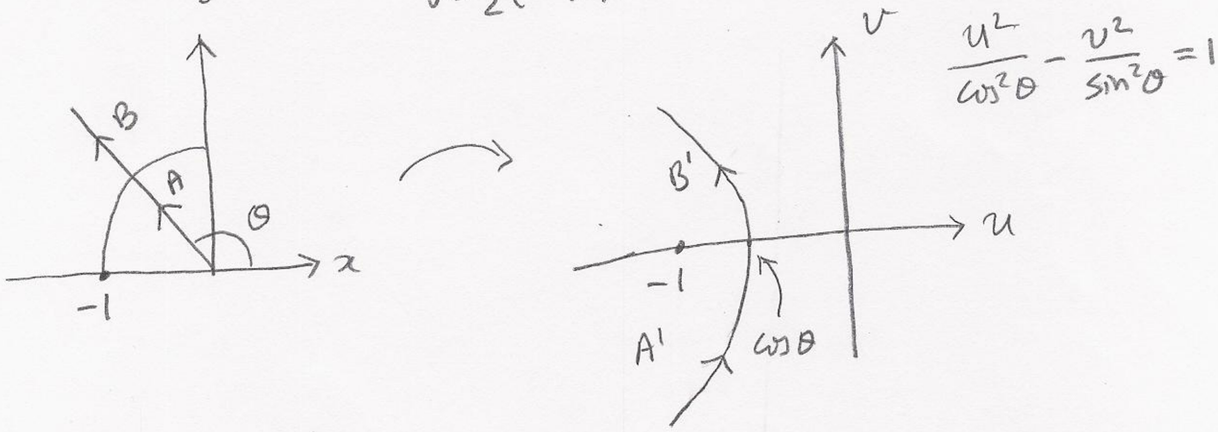






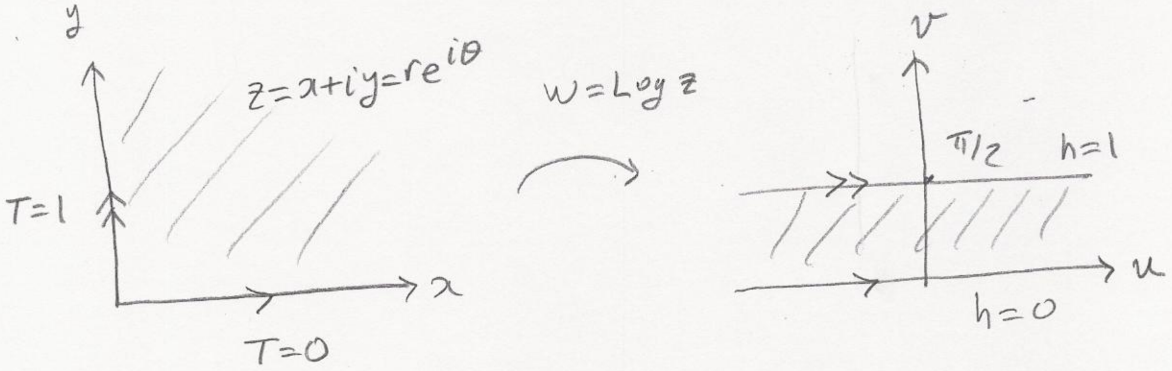
$u=\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta$

$v=\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta$



NAME: STUDENT NO:

Q-5) Find a bounded harmonic function $T(x, y)$ defined on the first quadrant, $0 \le x, y$, with the possible exception of the origin, such that $T(x, 0) = 0$ and $T(0, y) = 1$.



$$w = \operatorname{Log} z = \ln r + i\theta = u + iv$$

The function $h(u,v) = \frac{2}{\pi} v$ is harmonic and satisfies the boundary conditions

$$h(u,0) = 0, \quad h(u, \tfrac{\pi}{2}) = 1.$$

Pulling this function back to our original setting we get

$$T(x,y) = \tilde{T}(r,\theta) = h(\ln r, \theta) = \frac{2}{\pi}\theta = \frac{2}{\pi}\arctan\frac{y}{x}.$$

so $T(x,y) = \frac{2}{\pi} \arctan \frac{y}{x}$ is the required function.

Date: 30 May, 2003, Friday
Instructor: Ali Sinan Sertöz
Time: 9:00-11:00

Math 206 Complex Calculus – Final Exam
Solutions

1 Solve the following differential equation using Laplace transform techniques:

$$f''(t) - 3f'(t) + 2f(t) = e^{3t}$$

where $f(0) = 0$, $f'(0) = 1$.

Solution: We apply Laplace transform to both sides of the differential equation using the formulas

$$\begin{aligned}
\mathfrak{L}(f(t)) &= F(s), \\
\mathfrak{L}(f'(t)) &= sF(s) - f(0) \\
&= sF(s), \\
\mathfrak{L}(f''(t)) &= s^2F(s) - sf(0) - f'(0) \\
&= s^2F(s) - 1, \\
\mathfrak{L}(e^{3t}) &= \frac{1}{s-3}.
\end{aligned}$$

The equation then becomes

$$(s^2 - 3s + 2)F(s) - 1 = \frac{1}{s-3}.$$

Note that $s^2 - 3s + 2 = (s-1)(s-2)$. Solving for $F(s)$ we find that

$$\begin{aligned}
F(s) &= \frac{1}{(s-1)(s-2)}\left(\frac{1}{s-3} - 1\right) \\
&= \frac{1}{(s-1)(s-2)}\left(\frac{s-2}{s-3}\right) \\
&= \frac{1}{(s-1)(s-3)} \\
&= -\frac{1}{2}\frac{1}{(s-1)} + \frac{1}{2}\frac{1}{(s-3)}.
\end{aligned}$$

We easily take Laplace inverse transform of both sides of this equation and get

$$f(t) = -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t}.$$

---

2 Calculate all values of $(-4)^{-3/2}$, and indicate the principal value.

Solution:

$$\begin{aligned}(-4)^{-3/2} &= \exp\left(-\frac{3}{2}\log(-4)\right)\\ &= \exp\left(-\frac{3}{2}[\ln 4 + i(2n+1)\pi]\right)\\ &= \exp\left(\ln\frac{1}{8} - i\frac{3}{2}(2n+1)\pi\right)\\ &= \frac{1}{8}\left(\cos\frac{3}{2}(2n+1)\pi - i\sin\frac{3}{2}(2n+1)\pi\right), \quad n\in\mathbb{Z}.\end{aligned}$$

The principal value is obtained when $n=0$:

$$\begin{aligned}(-4)^{-3/2} &= \frac{1}{8}\left(\cos\frac{3}{2}\pi - i\sin\frac{3}{2}\pi\right)\\ &= \frac{i}{8}.\end{aligned}$$

---

3) Evaluate the integral $\int\limits_0^\infty \frac{x^{1/5}}{1+x^5}dx.$

Solution:

We integrate the function $f(z) = \frac{z^{1/5}}{1+z^5} = \frac{\exp(\frac{1}{5}\log z)}{1+z^5}$ around the closed contour $P_{\rho,R} = C_R - L_{\rho,R} - C_\rho + [\rho, R]$ where $R>1$, $0<\rho,1$ and

$$\begin{aligned}C_R &= \{Re^{i\theta} \mid 0\le\theta\le 2\pi/5\},\\ L_{\rho,R} &= \{xe^{2\pi/5} \mid \rho\le x\le R\},\\ C_\rho &= \{\rho e^{i\theta} \mid 0\le\theta\le 2\pi/5\},\\ [\rho,R] &= \{x \mid \rho\le x\le R\}.\end{aligned}$$

Inside this contour there is only one pole of $f(z)$, which is $z=e^{i\pi/5}$. By the residue theorem we have

$$\begin{aligned}\int_{P_{\rho,R}} f(z)dz &= 2\pi i \operatorname*{Res}_{z=e^{i\pi/5}} f(z)\\ &= 2\pi i\left(\frac{z^{1/5}}{5z^4}\right)_{z=e^{i\pi/5}}\\ &= \frac{2}{5}\pi i\left(z^{-19/5}|_{z=e^{i\pi/5}}\right)\\ &= \frac{2}{5}\pi i\left(e^{-\frac{19}{25}\pi i}\right)\\ &= -\frac{2}{5}\pi i e^{-\frac{6}{25}\pi i}\\ &= -\frac{2}{5}\pi i\left(\cos\frac{6}{25}\pi + i\sin\frac{6}{25}\pi\right)\\ &= \frac{2}{5}\sin\frac{6}{25}\pi - i\frac{2\pi}{5}\cos\frac{6}{25}\pi.\end{aligned}$$

On C_R we have:

$$\left|\int_{C_R} f(z)dz\right| \le \frac{R^{1/5}}{R^5-1}\frac{2\pi R}{5} \to 0 \text{ as } R\to\infty.$$

On C_ρ we have:

$$\left|\int_{C_\rho} f(z)dz\right| \leq \frac{\rho^{1/5}}{1-\rho^5}\frac{2\pi\rho}{5} \to 0 \text{ as } \rho \to 0.$$

Moreover on $L_{\rho,R}$ we have $z = xe^{2\pi i/5}$, $f(z)dz = f(x)e^{12\pi i/25}dx$ and hence

$$\int_{L_{\rho,R}} f(z)dz = e^{12\pi i/25}\int_\rho^R f(x)dx \to e^{12\pi i/5}\int_0^\infty \frac{x^{1/5}}{1+x^5}dx \text{ as } R\to\infty,\ \rho\to 0.$$

and

$$\int_{P_{\rho,R}} f(z)dz \to \left(1-e^{12\pi i/25}\right)\int_0^\infty \frac{x^{1/5}}{1+x^5}dx \text{ as } R\to\infty,\ \rho\to 0.$$

Combining this with the residue calculation above, we find

$$\left[(1-\cos\frac{12\pi i}{25}) - i\sin\frac{12\pi i}{25}\right]\int_0^\infty \frac{x^{1/5}}{1+x^5}dx \quad = \quad \frac{2}{5}\sin\frac{6}{25}\pi - i\frac{2\pi}{5}\cos\frac{6}{25}\pi$$

which gives

$$\int_0^\infty \frac{x^{1/5}}{1+x^5}dx = \frac{2\pi}{5}\frac{\cos\frac{6\pi}{25}}{\sin\frac{12\pi}{25}} = \frac{\pi}{5}\frac{1}{\sin\frac{6\pi}{25}} = 0.91786...$$

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4) Consider the linear fractional transformation $f(z) = \dfrac{z-1}{z+1}$, and describe the images of the following sets under f.
i) The upper half plane.
ii) The unit circle.
iii) The x-axis.
iv) The y-axis.

Solution:
First write $f(z)$ in terms of x and y:

$$f(z) = \frac{z-1}{z+1} = \frac{x-1+iy}{x+1+iy} = \frac{x^2+y^2-1}{(x+1)^2+y^2} + i\frac{2y}{(x+1)^2+y^2} = u+iv.$$

i) When $y \geq 0$, we have $v \geq 0$. Hence the upper half plane maps onto the upper half plane. Here we used the fact that a nonconstant linear fractional transformation is onto.

ii) When $x^2+y^2 = 1$, we have $u = 0$. Hence the unit circle maps onto the v-axis.

iii) Note that $f(-1) = \infty$, $f(0) = 1$ and $f(1) = 0$. The x-axis, which is a circle, must map onto the circle which passes through the points ∞, -1 and 1. Hence the image is the u-axis.

iv) Note again that $f(0) = -1$, $f(i) = i$ and $f(\infty) = 1$. The y-axis, which is a circle, must map onto the circle which passes through the points -1, i and 1. Hence the image is the unit circle.

Date: 29 May 2002, Wednesday Instructor: Ali Sinan Sertöz

Math 206 Complex Calculus – Final Exam
Solutions

Q-1) Solve the following recursion equation:

$$f(n+2) - 7f(n+1) + 12f(n) = 2^n,\ f(0) = f(1) = 0.$$

Solution: Using $\mathcal{Z}$-transformation we recall that

$$\begin{aligned}
\mathcal{Z}(f(n)) &= F(z), \\
\mathcal{Z}(f(n+1) &= zF(z) - zf(0) = zF(z), \\
\mathcal{Z}(f(n+2) &= z^2F(z) - z^2f(0) - zf(1) = z^2F(z), \\
\mathcal{Z}(2^n) &= \frac{z}{z-2}.
\end{aligned}$$

Taking the $\mathcal{Z}$-transform of both sides of the equation we get

$$\begin{aligned}
(z^2 - 7z + 12)F(z) &= \frac{z}{z-2}, \quad \text{or} \\
F(z) &= \frac{z}{(z-2)(z-3)(z-4)}.
\end{aligned}$$

Recalling that under the $\mathcal{Z}$-transform most functions go to a fraction with a z in the numerator, we use the partial fractions technique as follows;

$$\begin{aligned}
F(z) &= \frac{z}{(z-2)(z-3)(z-4)} \\
&= z\left[\frac{1}{(z-2)(z-3)(z-4)}\right] \\
&= z\left[\frac{1}{2}\frac{1}{z-2} - \frac{1}{z-3} + \frac{1}{2}\frac{1}{z-4}\right] \\
&= \frac{1}{2}\frac{z}{z-2} - \frac{z}{z-3} + \frac{1}{2}\frac{z}{z-4}
\end{aligned}$$

Taking inverse $\mathcal{Z}$-transform now gives

$$f(n) = (\frac{1}{2})2^n - 3^n + (\frac{1}{2})4^n,$$

or after simplifying,

$$f(n) = 2^{n-1} + 2^{2n-1} - 3^n,\ n = 0, 1, ...$$

You should in the exam check that the answer you find is actually a solution of the given equation.

Q-2) Let R be the region defined as

$$R = \{\, z \in \mathbb{C} \mid 1 \leq |z| \leq 2,\ \operatorname{Im} z \geq 0 \,\}$$

Consider the transformation $f(z) = z + \dfrac{1}{z}$.
Describe $f(R)$.
Describe the image of the boundary of R.
Is the transformation conformal?

Solution: This map is studied on page 374 of your book.
Let $z = re^{i\theta}$. Then

$$f(z) = u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

If $r = 1$, then $w = 2\cos\theta$ and the inner circle maps onto the real interval $[-2, 2]$ in the w plane.
If $1 < r \leq 2$, then $r - 1/r \neq 0$ and we obtain

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1.$$

If $0 \leq \theta \leq \pi$ and $r > 1$, then $v > 0$, so we get the part of this ellipse which is in the upper half plane. Putting $r = 2$ we find the outermost ellipse in the image. The interior points of R are mapped to the interior points of this outermost ellipse with $v > 0$.
The outer circle, $r = 2$, maps onto this outermost ellipse.
When $\theta = 0$, f maps $[1, 2]$ onto $[2, 5/2]$.
When $\theta = \pi$, f maps $[-2, -1]$ onto $[-5/2, -2]$.
$f'(z) = 1 - 1/z^2 = 0$ only at $z = \pm 1$, so f is conformal at every other point.

Q-3) Solve the following boundary value problem for a bounded T;

$$\begin{aligned} T_{xx}(x,y) + T_{yy}(x,y) &= 0, \quad y \geq 0,\ -\infty < x < \infty, \\ T(x,0) &= 0, \quad x < -2, \\ T(x,0) &= 1, \quad x > 2, \\ T_y(x,0) &= 0, \quad -2 < x < 2. \end{aligned}$$

Solution: This is *almost* Exercise 6 on page 308, and the solution uses exactly the same argument given on page 306.
Consider the region R given in the w plane by $v \geq 0$ and $-\pi/2 \leq u \leq \pi/2$. The map $z = 2\sin w$ sends this region onto our region, conformally except at the points $u = \pm\pi/2$. A solution to our problem in R is $T(u, v) = (1/2) + (1/\pi)u$. Check that it is a solution.
$z = 2\sin w$ becomes $x + iy = 2\sin u \cosh v + i2\cos u \sinh v$. Eliminating v we get

$$\frac{x^2}{4\sin^2 u} - \frac{y^2}{4\cos^2 u} = 1.$$

Using the properties of hyperbolas, this gives

$$4\sin u = \sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2}$$

and solving for u finally gives

$$T(x,y) = \frac{1}{2} + \frac{1}{\pi}\arcsin\left[\frac{1}{4}\left(\sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2}\right)\right],$$

where $-\pi/2 \leq \arcsin t \leq \pi/2$ since this is the range for u.

Q-4) Describe the image of the x-axis under the Schwarz-Christoffel transformation

$$f(z) = \alpha \int_0^z (s^2 - 1)^{-3/4}\, s^{-1/2} ds, \quad \text{where } \alpha = e^{i3\pi/4}.$$

Hint: $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p,q > 0$, is the Beta function and in particular $B(1/4,1/4) = 7.416...$

Solution: This is a reformulation of Exercise 1 on page 336.
We can set $x_1 = -1$, $x_2 = 0$, $x_3 = 1$. The corresponding constants describing the angles are $k_1 = 3/4$, $k_2 = 1/2$, $k_3 = 3/4$. Since $k_1 + k_2 + k_3 = 2$, the image is a triangle. Since one of the angles is $k_2\pi = \pi/2$, this is a right triangle. Since $k_1 = k_3$, this is an isosceles right triangle. $f(0) = 0$ is the right angle vertex of the triangle. To find $f(1)$ we evaluate the integral:

$$f(1) = \alpha \int_0^1 (s^2 - 1)^{-3/4}\, s^{-1/2} ds,$$

but here the $(s^2 - 1)$ factor is negative and a fourth root of it will be imaginary. We write it as

$$\begin{aligned}(s^2-1)^{-3/4} &= (-1)^{-3/4}(1-s^2)^{-3/4} \\ &= \alpha^{-1}(1-s^2)^{-3/4}\end{aligned}$$

and the integral becomes

$$f(1) = \int_0^1 (1 - s^2)^{-3/4}\, s^{-1/2} ds$$

which is a real integral. Say $f(1) = b \in \mathbb{R}^+$. Writing the integral for $f(-1)$ and making the substitution $t = -s$ we obtain that $f(-1) = if(1) = ib$. Furthermore making the substitution $t = s^2$ in the integral for $f(1)$ we find that $b = (1/2)B(1/4, 1/4)$.
Thus the real line maps onto the isosceles right triangle with right vertex at the origin and the other vertices at $(b, 0)$ and $(0, ib)$.