Date: November 4, 2004, Thursday
Instructor: Özgüler
Time: 17.30-19.30

NAME:..

STUDENT NO:..

Math 206 Complex Calculus–Midterm 1

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet.
Write your name on the top of every page.

Q-1) Find all solutions of the equation $z^4 + 16i = 0$ in polar coordinates and mark them on the complex plane.

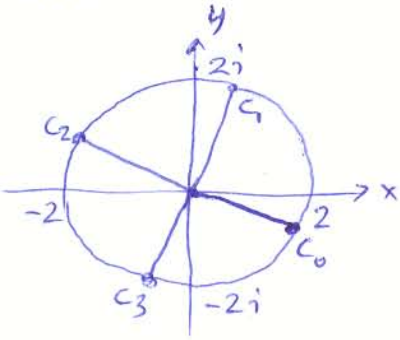
$$-16i = 2^4 e^{i(-\frac{\pi}{2}+2k\pi)}, \quad k \in \mathbb{Z}$$

$$(-16i)^{1/4} = 2e^{i(-\frac{\pi}{8}+\frac{k\pi}{2})}, \quad k = 0,1,2,3$$

Thus,

$$c_0 = 2e^{-i\pi/8}, \quad c_1 = 2e^{i3\pi/8}, \quad c_2 = 2e^{i7\pi/8}, \quad c_3 = 2e^{i\frac{11\pi}{8}}$$

are the solutions.



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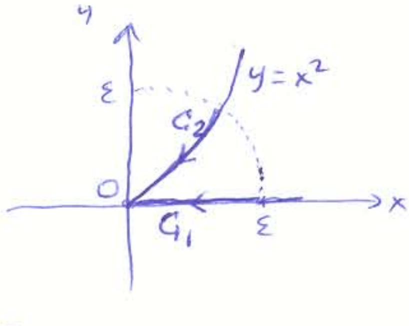
Q-2) Let

$$f(z) = \begin{cases} \dfrac{x^2y(y-ix)}{x^4+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

i) Show that $f(z)$ is not differentiable at $z = 0$.
ii) Identify the region of the complex plane in which $f(z)$ is differentiable. Is it a domain?



i) Consider two possible ways of z approaching the origin O:
Along C_1: $z = x$, $0 < x \leq \varepsilon$
Along C_2: $z = x + ix^2$, $0 < x \leq \varepsilon$
where ε is a positive real number.

Along C_1, $y = 0$, so that

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \left.\frac{x^2y\,(y-ix)}{(x^4+y^2)(x+iy)}\right|_{y=0} = \lim_{x\to 0} 0 = 0.$$

Along C_2, $y = x^2$, so that

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{x^4(x^2-ix)}{2x^4(x+ix^2)} = \lim_{x\to 0}\left(-i\,\frac{1}{2}\right) = \frac{-i}{2}$$

Since the two limit values differ, $f'(0)$ does not exist.

ii) If $z \neq 0$, then the real and imaginary parts u and v of f are both rational functions of x, y. Thus, the first order partial derivatives of u and v exist and are continuous everywhere except $z = 0$. We check the Cauchy-Riemann eq's:
$u = x^2y^2/(y^2+x^4)$, $v = -x^3y/(y^2+x^4)$ so that

$u_x - v_y = x(y^2-x^4)(2y^2-x^2)/(y^2+x^4)$, $u_y + v_x = -3x^2y\,(y^2-x^4)/(y^2+x^4)$.

Both are satisfied if and only if "$x = 0$ or $y^2 = x^4$". Therefore, <u>$f'(z)$ exists if and only if $z \neq 0$ is either on the real axis or on one of the parabolas $y = \mp x^2$.</u> This is <u>not</u> a domain since no point on any curve has a neighborhood contained in the union of these curves.

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Q-3) Let $a = \frac{e}{\sqrt{2}}(1-i)$ and $b = i\pi$. Find the principal value of a^b.

By definition, $a^b := \exp[b \log a]$. The principal value is $\exp[b \operatorname{Log} a] = \exp[i\pi (\ln|a| + i \operatorname{Arg} a)]$.

Hence,

$$\exp[b \operatorname{Log} a] = \exp[i\pi (\ln e + i(-\pi/4))]$$
$$= \exp\left[\frac{\pi^2}{4} + i\pi\right]$$
$$= \exp\left[\frac{\pi^2}{4}\right] \exp(i\pi)$$
$$= \boxed{-e^{\pi^2/4}}$$

Q-4) Find all solutions of the equation $\sin z = i$ by:
i) Using the expression $\sin z = \sin x \cosh y + i \cos x \sinh y$.
ii) Using the inverse function $\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]$.

i) $\sin x \cosh y + i \cos x \sinh y = i \Rightarrow \sin x \cosh y = 0,$
$\cos x \sinh y = 1.$

Since $\cosh y \neq 0$ for any y, the first equality gives $\sin x = 0$. Hence, $\boxed{x = n\pi, \; n \in \mathbb{Z}}$. If n is even, then, in the second equality, $\cos x = \cos(n\pi) = 1$, $\sinh y = 1$.

Thus,

$$\sinh y = \frac{e^y - e^{-y}}{2} = 1 \Rightarrow e^{2y} - 2e^y - 1 = 0$$

$$\Rightarrow e^y = 1 \mp \sqrt{2} \Rightarrow e^y = 1 + \sqrt{2} \quad (\text{since } 1 - \sqrt{2} < 0)$$

$$\Rightarrow \underline{y = \ln(1 + \sqrt{2})}.$$

If n is odd, then $\cos x = -1$, $\sinh y = -1$ so that $e^{2y} + 2e^y - 1 = 0$, which gives $e^y = -1 \mp \sqrt{2}$ or $\underline{e^y = \sqrt{2} - 1}$.

Therefore, all solutions are

$$z = \begin{cases} n\pi + i \ln(1 + \sqrt{2}) & , n \text{ even} \\ n\pi + i \ln(\sqrt{2} - 1) & , n \text{ odd}. \end{cases}$$

Noting that $\ln(\sqrt{2} - 1) = -\ln(1 + \sqrt{2})$, we can express all solutions by the single formula:

$$\boxed{z = n\pi + i(-1)^n \ln(1 + \sqrt{2}), \; n \in \mathbb{Z}}$$

ii) $\sin^{-1} i = -i \log\left[i^2 + (1+1)^{1/2}\right] = -i \log(-1 \mp \sqrt{2}).$

$$= \begin{cases} -i\left[\ln(1+\sqrt{2}) + i n\pi\right], n \text{ odd} & (\text{if `}-\text{' is chosen}) \\ -i\left[\ln(\sqrt{2}-1) + i n\pi\right], n \text{ even} & (\text{if `}+\text{' is chosen}) \end{cases}$$

$$= \begin{cases} n\pi - i \ln(1+\sqrt{2}) & , n \text{ odd} \quad (\text{for `}-\text{' sign}) \\ n\pi - i \ln(\sqrt{2}-1) & , n \text{ even} \quad (\text{for `}+\text{' sign}) \end{cases}$$

Hence, noting $\ln(\sqrt{2}-1) = -\ln(1+\sqrt{2})$, all solutions are

$$\boxed{z = n\pi + i(-1)^n \ln(1+\sqrt{2}), \; n \in \mathbb{Z}}$$

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Q-5) Let C be the positively oriented unit circle at the origin. Evaluate

$$\int_C f(z)dz$$

when $f(z)$ is the principal branch of

$$z^{-1+i} = exp[(-1+i)\,log\,z].$$

The principal branch of z^{-1+i} is

$$\exp[(-1+i)\operatorname{Log} z], \quad (|z|>0,\ -\pi<\arg z<\pi)$$

A parametric description of the unit circle compatible with this branch is

$$C:\ z=e^{i\theta}, \quad -\pi\le\theta\le\pi.$$

Hence,

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} \exp[(i-1)(\ln|e^{i\theta}|+i\theta)]\,(ie^{i\theta})\,d\theta$$

$$= \int_{-\pi}^{\pi} \exp[(i-1)(0+i\theta)]\; i e^{i\theta}\,d\theta$$

$$= \int_{-\pi}^{\pi} e^{-\theta} e^{-i\theta}\, i e^{i\theta}\, d\theta$$

$$= i\int_{-\pi}^{\pi} e^{-\theta}\,d\theta = i\,(-e^{-\theta})\Big|_{-\pi}^{\pi}$$

$$= i(e^{\pi}-e^{-\pi}) = \boxed{2i\sinh\pi}.$$