Date: October 31, 2006, Tuesday NAME:..
Time: 17:40-19:10

STUDENT NO:...

Math 206 Complex Calculus – Midterm Exam I

PLEASE READ:

This is a closed-book exam. Check that there are 4 questions on your exam booklet. No correct answer without a satisfying reasoning is accepted. Show your work in detail. Write your name on the top of every page.

Q-1) Show that $\lim_{z\to 0} (\frac{z}{\bar{z}})^2$ does not exist.

If we approach the origin along the x-axis, i.e., if we restrict $z = x$, then we get

$$\lim_{z\to 0} (\frac{z}{\bar{z}})^2 = \lim_{y=0,\, x\to 0} (\frac{x^2 - y^2 + i2xy}{x^2 + y^2})^2 = 1.$$

whereas, if we approach the origin along the line $x = y$, then we get

$$\lim_{z\to 0} (\frac{z}{\bar{z}})^2 = \lim_{x=y,\, x\to 0} (\frac{x^2 - y^2 + i2xy}{x^2 + y^2})^2 = i^2 = -1.$$

Since the two limits differ, and since whenever limit exists, it is unique, we conclude that $\lim_{z\to 0} (\frac{z}{\bar{z}})^2$ does not exist.

Q-2) (i) Show that $u(x,y) = sinh(x)\ sin(y)$ is harmonic in the entire plane and (ii) find *all* harmonic conjugates $v(x,y)$ of $u(x,y)$.

(i) The function u has continuous partial derivatives of all orders in the entire plane because $sinh(x)$ and $sin(y)$ each have. We also note

$$u_x = cosh(x)\,sin(y),\ u_{xx} = sinh(x)\,sin(y),$$

$$u_y = sinh(x)\,cos(y),\ u_{yy} = -sinh(x)\,sin(y),$$

in the entire plane, which shows that Laplace's equation holds. Therefore u is harmonic in the whole complex plane. (ii) By Cauchy-Riemann equations $v_y = u_x = cosh(x)\,sin(y)$ so that $v(x,y) = -cosh(x)\,cos(y) + \phi(y)$ for some function ϕ that depends only on the variable y. Also by C-R equations, $v_x = -sinh(x)\,cos(y) + \phi'(y) = -u_y = -cosh(x)\,sin(y)$. This gives that $\phi'(y) = 0$, or that $\phi(y) = c$ for some real constant c. Therefore, necessarily $v(x,y) = -cosh(x)\,cos(y) + c$ for some arbitrary c. It is now easy to check that each $v(x,y)$ in this form satisfies C-R equations and hence all harmonic conjugates are in this form.

Q-3) Find *all* $z = x + iy$ satisfying the equality

$$cos(z) + i = 0.$$

Note: Give your answers in terms of ln (the real natural logarithmic function)!

By the identity $cos(z) = cos(x)cosh(y) - isin(x)sinh(y)$, the equality holds if and only if $cos(x)cosh(y) = 0$ and $sin(x)sinh(y = 1$, simultaneously. Since $cosh(y) \neq 0$ for every real y, the first equality holds if and only if $x = (2k+1)\pi/2$ for some integer k. Then, for any such x, $sin(x) \neq 0$ and the second equality holds if and only if $sinh(y) = (e^y - e^{-y})/2 = \pm 1$, or equivalently, $e^{2y} - 2e^y - \pm 1 = 0$, where the sign is positive if $x = (4n+1)\pi/2$ and negative if $x = (4n+3)\pi/2$. Solving this equation for both signs, $e^y = \pm 1 \pm \sqrt{2}$. Since e^y must be positive, we get $e^y = 1 + \sqrt{2}$, or $e^y = -1 + \sqrt{2}$ which give $y = \pm ln((1 + \sqrt{2})$. Therefore, combining the two cases, all solutions are

$$z = x + iy = \frac{(2n+1)\pi}{2} + i\,(-1)^n\, ln(1 + \sqrt{2}), \ n \text{ integer.}$$

Q-4) Show that

$$\lim_{R\to\infty} \int_{C_R} \frac{Log\, z}{z^2}\, dz = 0$$

when C_R is the positively oriented semi-circular contour in the upper plane, i.e.,

$$C_R : z = R\, e^{i\theta}, \; -\pi \leq \theta \leq \pi.$$

We use the inequality

$$|\int_C f(z)\, dz| \leq M\, L,$$

where M is such that $|f(z)| \leq M$ at every $z \in C$ and L is the length of C. Now, for each $z = Re^{i\theta} \in C_R$, we have

$$|\frac{Log\, z}{z^2}| = |\frac{ln\, R + i\theta}{R^2\, e^{i2\theta}}| = \frac{|ln\, R + i\theta|}{R^2} \leq \frac{\sqrt{(ln\, R)^2 + \pi^2}}{R^2} \leq \frac{ln\, R + \pi}{R^2} =: M$$

so that, since C_R has length $\pi\, R$, we obtain

$$|\int_C f(z)\, dz| \leq \pi \frac{ln\, R + \pi}{R}.$$

The right hand side, by L'Hopital's rule, tends to zero as R tends to infinity. Therefore, the left hand side, and hence what is inside the modulus, have also limits zero as R tends to infinity.