Date: December 1, 2007, Saturday NAME:..

Time: 13:30-15:30

Özgüler STUDENT NO:..

Math 206 Complex Calculus – Midterm Exam II

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:

Check that there are 4 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) Determine two Laurent series expansions for

$$f(z) = \frac{1}{z(1+z^2)}$$

about $z_0 = 0$ and in each case indicate the region in which the expansion is valid.

Solution: The two annulus-like regions are $R_1 : \ 0 < |z| < 1$ and $R_2 : \ 1 < |z| < \infty$. For $z \in R_1$, we have the Laurent series

$$f(z) = \frac{1}{z} \sum_{n=0}^{n=\infty} (-1)^n \, z^{2n} = \sum_{n=0}^{n=\infty} (-1)^n \, z^{2n-1} \sum_{k=0}^{k=\infty} (-1)^{k+1} \, z^{2k+1} + \frac{1}{z}$$

and for $z \in R_2$, the Laurent series

$$f(z) = \frac{1}{z^3} \sum_{n=0}^{n=\infty} (-1)^n \, z^{-2n} = \sum_{n=0}^{n=\infty} (-1)^n \, z^{-2n-3} = \sum_{k=1}^{k=\infty} (-1)^{k-1} \, z^{-2k-1}.$$

Q-2) Find all roots of

$$sin\, z - cos\, z = 3.$$

Solution: We have

$$sinz - cosz = cosh\, y\ (sin\, x - cos\, x) + i[sinh\, y\ (sin\, x + cos\, x)] = 3$$

so that

$$cosh\, y\ (sin\, x - cos\, x) = 3,\ sinh\, y\ (sin\, x + cos\, x) = 0.$$

The second equality holds if and only if $sin\, x = -cos\, x$ since $sinh\, y = 0$ would imply that $y = 0$ and $sin\, x - cos\, x = 3$ which is not possible. Hence, $sin\, z - cos\, z = 3$ if and only if

$$cosh\, y\ sin\, x = \frac{3}{2} \text{ and } sin\, x = -cos\, x.$$

Noting from the first equality that $sin\, x > 0$, this in turn holds if and only if

$$x = \frac{3\pi}{4} + 2n\pi \text{ and } cosh\, y = \frac{3}{\sqrt{2}}$$

or equivalently

$$x = \frac{3\pi}{4} + 2n\pi \text{ and } e^y + e^{-y} = 3\sqrt{2}.$$

Solving the quadratic equation that results for e^x, we obtain

$$x = \frac{3\pi}{4} + 2n\pi \text{ and } ln\, (\frac{3 \pm \sqrt{7}}{\sqrt{2}}),$$

or

$$z = \frac{3\pi}{4} + 2n\pi + i\ ln\, (\frac{3 \pm \sqrt{7}}{\sqrt{2}}),$$

for an arbitrary integer n

Q-3) Evaluate the integral

$$\int_C \frac{e^z}{(z-a)(z-b)^2}\, dz$$

if C is a simple closed contour containing both of the singular points $0 < a < b$ on the positive real axis.
HINT: You can either use the Residue Theorem or use the Cauchy Integral Formula *twice* by dividing the contour C into C_1 and C_2 such that each contain exactly one singular point and $C = C_1 + C_2$.

Solution: If C_1 and C_2 are chosen as rectangular closed contours enclosing a and b, respectively, and their one edge coinciding between points a and b, then $C = C_1 + C_2$. By Cauchy Integral Formula

$$\int_{C_1} \frac{e^z}{(z-a)(z-b)^2}\, dz = i2\pi\left(\frac{e^z}{(z-b)^2}\right)\Big|_{z=a} = i2\pi\frac{e^a}{(a-b)^2}.$$

Similarly, by the generalized Cauchy Integral Formula

$$\int_{C_2} \frac{e^z}{(z-a)(z-b)^2}\, dz = i2\pi\frac{d}{dz}\left(\frac{e^z}{z-a}\right)\Big|_{z=b} = i2\pi\frac{e^b(b-a-1)}{(a-b)^2}.$$

Hence,

$$\int_C \frac{e^z}{(z-a)(z-b)^2}\, dz = i2\pi\frac{e^a + e^b(b-a-1)}{(a-b)^2}.$$

Q-4) Evaluate the integral

$$\int_C \frac{\bar{z}}{z}\, dz$$

if C is the upper half of the unit circle extending from $z = 1$ to $z = -1$.

Solution: A parametrization for the contour C is

$$C : z = e^{i\theta},\ 0 \leq \theta < \pi.$$

Thus,

$$\int_C \frac{\bar{z}}{z}\, dz = \int_0^{\pi} \frac{e^{-i\theta}}{e^{i\theta}}\, i\, e^{i\theta}\, d\theta = i \int_0^{\pi} e^{-i\theta}\, d\theta = -(e^{-i\pi} - e^{-i0}) = 2.$$