Date: November 21, 2006, Tuesday NAME:..

Time: 10:40-12:10

STUDENT NO:..

Math 206 Complex Calculus – Midterm Exam II

PLEASE READ:

This is a closed-book exam. Check that there are 4 questions on your exam booklet. No correct answer without a satisfying reasoning is accepted. Show your work in detail. Write your name on the top of every page.

Q-1) Find *two different Laurent series expansions* for the function

$$f(z) = \frac{1}{z^3(1-z)^2}$$

and *indicate the regions* where each expansion is valid.

Solution: $f(z)$ is analytic in two annular domains $D_1 : 0 < |z| < 1$ and $D_2 : 0 < |z-1| < 1$. We thus have the Laurent series expansion in D_1 as

$$f(z) = \frac{1}{z^3}\frac{d}{dz}\sum_{n=0}^{\infty} z^n = \frac{1}{z^3}\sum_{n=1}^{\infty} n\, z^{n-1} = \frac{1}{z^3} + \frac{2}{z^2} + \frac{3}{z} + \sum_{k=0}^{\infty}(k+4)z^k,$$

where we used the fact that $\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z}$. In D_2, we have

$$f(z) = \frac{1}{(1-z)^2}\frac{1}{[1-(1-z)]^3}.$$

Now,

$$\frac{d^2}{dw^2}\frac{1}{1-w} = \frac{2}{(1-w)^3},$$

so that

$$\frac{2}{[1-(1-z)]^3} = \frac{d^2}{dz^2}\sum_{n=0}^{\infty}(1-z)^n = \sum_{n=2}^{\infty} n(n-1)(1-z)^{n-2}.$$

This gives

$$f(z) = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(1-z)^{n-4} = \sum_{k=-2}^{\infty}(k+3)(k+4)(1-z)^k \quad (0 < |z-1| < 1).$$

Q-2) Determine the residue at every isolated singularity of the functions

$$(10pts.)\ (i) \qquad f(z) = \frac{1}{(z^3+3z^2+3z+1)^2},$$

$$(15pts.)\ (ii) \qquad g(z) = \frac{z-1}{z(z-2)}.$$

Solution: (i) The only (isolated) singularity of $f(z)$ is at $z=-1$ and

$$f(z) = \frac{1}{(z+1)^6}$$

is already in Laurent series form with region of validity $0 < |z+1| < \infty$. The residue is hence $b_1 = 0$. (ii) There are two isolated singularities of $g(z)$ at $z=0$ and $z=2$. To determine $Res_{z=0}g(z)$, we write

$$g(z) = (\frac{z-1}{z})(\frac{1}{2})(\frac{-1}{1-z/2}) = \frac{1}{2}(\frac{1}{z}-1)\sum_{n=0}^{\infty}\frac{z^n}{2^n} \quad (|z|<2),$$

so that $Res_{z=0}g(z) = 1/2$. To determine $Res_{z=2}g(z)$, we write

$$g(z) = (\frac{z-1}{z-2})(\frac{1}{2+(z-2)}) = (1+\frac{1}{z-2})\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{2^n} \ (|z-2|<2),$$

so that $Res_{z=2}g(z) = 1/2$.

Q-3) Use MacLaurin series expansions of $sin(z)$ and $\frac{1}{1+z}$ to determine the residue at $z = 0$ of

$$\frac{1}{(1+z)\, sin(z)}.$$

Solution: We have

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^z\, z^n = 1 - z + z^2 - z^3 + ... \quad (|z| < \infty),$$

$$sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{2} + \frac{z^5}{6} - ... \quad (|z| < \infty),$$

so that by long dividing the first by the second , we get

$$\frac{1}{(1+z)\, sin(z)} = \frac{1}{z} - 1 + \frac{3z}{2} - ... \quad (0 < |z| < \infty),$$

This gives $Res_{z=0} \frac{1}{(z+1)sin(z)} = 1$.

Q-4) Find the value of the integrals

$$(10pts.)\ (i)\ \int_C \frac{z\, e^z\, Log(z)}{(z-1)^3}\, dz,$$

where C is the positively oriented circle $|z-1| = 1/2$,

$$(15pts.)\ (ii)\ \int_C \frac{z^{99}}{2\, z^{100}+1}\, dz,$$

when C is the positively oriented circle $|z| = 1$.

Solution: (i) We use Cauchy's generalized integral formula with $z_0 = 1, n = 2$, and $f(z) = z\, e^z\, Log(z)$. Since $f^{(1)}(z) = z\, e^z\, Log(z) + e^z\, Log(z) + e^z$ and $f^{(2)}(z) = z\, e^z\, Log(z) + 2\, e^z\, Log(z) + 2e^z + z^{-1}\, e^z$, we obtain

$$\int_C \frac{z\, e^z\, Log(z)}{(z-1)^3}\, dz = \frac{i2\pi}{2!} f^{(2)}(1) = i3e\pi.$$

(ii) Here, we use the fact that $f(z) = \frac{z^{99}}{2\, z^{100}+1}$ is analytic outside the circle $|z| = 1$ and has a finite number of isolated sigularities inside it. Hence,

$$\int_C \frac{z^{99}}{2\, z^{100}+1}\, dz = i2\pi\ Res_{z=0}\ [\frac{1}{z^2} f(\frac{1}{z})].$$

Now

$$\frac{1}{z^2} f(\frac{1}{z}) = \frac{z^{100-99-2}}{2+z^{100}} = \frac{1}{z(2+z^{100})} = \frac{1}{2z} \sum_{n=0}^{\infty} (-1)^n \frac{z^{100n}}{2^n}, \quad (0 < |z| < 2),$$

and we get $Res_{z=0}\ [\frac{1}{z^2} f(\frac{1}{z})] = 1/2$, or,

$$\int_C \frac{z^{99}}{2\, z^{100}+1}\, dz = i\pi.$$