Von Neumann Entropy

For any density $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$,

$$S(\rho) = -\mathrm{tr}(\rho \log \rho)$$

or equivalently,

$$S(\rho) = \sum_i -\lambda_i \log \lambda_i.$$

Von Neumann entropy = Shannon entropy of eigenvalues.

Quantum relative entropy

For ρ and σ densities,

$$S(\rho||\sigma) = \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma)$$

Klein's inequality:

$$S(\rho||\sigma) \geq 0$$

with equality iff $\rho = \sigma$.

Properties of VN entropy

1) Nonnegativity.

$$S(\rho) \geq 0$$

with equality iff ρ is a pure state.

2) Maximum.

$$S(\rho) \leq \log d$$

where d is the dimension of ρ.

Equality iff $\rho = (1/d)I$.

3) Invariance. The entropy is unchanged under unitary transformations,

$$S(U\rho U^\dagger) = S(\rho).$$

4) Schmidt. For a composite system AB in a pure state,

$$S(\rho^A) = S(\rho^B).$$

5) Subadditivity. For a joint system AB,

$$S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B).$$

6) Concavity. For $\{p_i\}$ a p.d.,

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i),$$

with equality iff all ρ_i for which $p_i > 0$ are identical.

7) Measurements. Let $M = \sum_i \lambda_i P_i$ be a PM on ρ. Outcome λ_i with $p_i = \mathrm{tr}(P_i \rho)$.

$$H(p_i) \geq S(\rho)$$

8) State preparation. Let

$$\rho = \sum_x p_x |\psi_x\rangle\langle\psi_x|.$$

Then,

$$H(p_x) \geq S(\rho).$$

9) Strong subadditivity.

$$S(\rho^{ABC}) + S(\rho^{B}) \leq S(\rho^{AB}) + S(\rho^{BC}).$$

10) Araki-Lieb inequality.

$$S(\rho^{AB}) \geq |S(\rho^{A}) - S(\rho^{B})|.$$

N.B. $S(\rho^{AB}) \geq S(\rho^{A})$ is false in general.