The Discrete Fourier Transform (DFT)

A unitary transformation of complex vector space.

DFT: $x = (x_1, \ldots, x_N)^T \to y = (y_1, \ldots, y_N)^T$ such that

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi ijk/N}.$$

Inverse DFT (IDFT) is the transformation

$$x_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} y_k e^{-2\pi ijk/N}.$$

Example: DFT for $N = 8$

$$W_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

where $\omega = e^{2\pi i/8} = \sqrt{i}$.

The Quantum Fourier Transform (QFT)

Consider a quantum system with basis states $|0\rangle, \ldots, |N-1\rangle$.

QFT acts on basis vectors as follows:

$$|j\rangle \to \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle.$$

By superposition principle, QFT of an arbitrary state is

$$\sum_{j=0}^{N-1} x_j |j\rangle \to \sum_{k=0}^{N-1} y_k |k\rangle,$$

where $\{y_k\}$ is the DFT of $\{x_j\}$.

The QFT is a unitary operation and hence can be implemented using quantum gates.

Product Representation of QFT

Consider an n-qubit system with dimension $N = 2^n$.

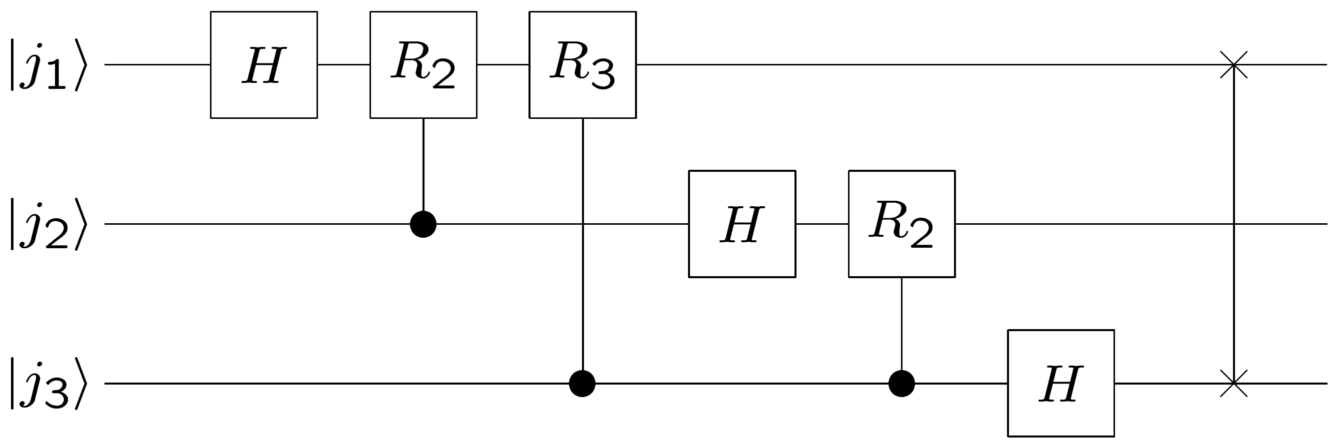
QFT: $|j\rangle = |j_1, \ldots, j_n\rangle \rightarrow$

$$\frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle) \cdots (|0\rangle + e^{2\pi i 0.j_1 \cdots j_n}|1\rangle)}{2^{n/2}}$$

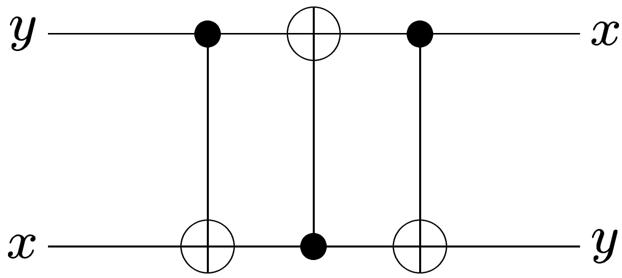
This operation can be implemented using only Hadamard, CNOT and $C(R_k)$ gates where

$$R_k = \left[\begin{array}{cc} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{array} \right].$$

Example: Three Qubit QFT



Swap Operation



Gate count for QFT

- n Hadamard gates
- $\binom{n}{2}$ controlled R_k gates
- At most $n/2$ swaps each requiring 3 CNOT gates

Total: $O(n^2)$ gates.

This is better than classical complexity of DFT which has circuit complexity $N\log(N)$ where $N = 2^n$ is the dimension.

See Nielsen and Chuang for a discussion.

Approximate QFT

Fact: If each R_k gate is approximated to precision $\Delta = 1/p(n)$ for some polynomial $p(n)$, the error in the QFT scales as $n^2/p(n)$.

Conclusion: QFT can be implemented using a number of elementary gates (H, T, and CNOT) bounded polynomially in n, the number of qubits.