Chapter 4

Quantum Theory Background

4.1 Quantum measurements

We will talk about three types of quantum measurements:

- Projective measurements (PM),
- Generalized measurements (GM),
- Positive-operator-valued-measure (POVM) measurements, also called Von Neumann measurements.

All these types of measurements turn out to be equivalent. POVM formalism is the most convenient one for our purposes.

4.1.1 Observables and projective measurements

An observable M is a Hermitian operator. Every observable M has a spectral decomposition

$$M = \sum_m \lambda_m P_m$$

where λ_m is an eigenvalue and P_m is the projector onto the eigenspace with eigenvalue λ_m. Note that

- λ_m are real-valued.
- P_m are Hermitian.
- Orthogonality: $P_m P_n = \delta_{nm} P_m$.

- Resolution of identity: $\sum_m P_m = I$.

Axiom: Every observable M defines a PM. A state $|\psi\rangle$ subjected to PM by observable

$$M = \sum_m \lambda_m P_m$$

goes into state

$$\frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}}$$

with probability

$$p(m) = \langle\psi|P_m|\psi\rangle.$$

The eigenvalue λ_m is registered as the measured value.

Repeatability: If the system is subjected to the same measurement immediately after a PM, the same outcome occurs with certainty.

The expectation of the measured value is

$$\begin{aligned} \langle M\rangle &= \sum_m \lambda_m p(m) \\ &= \sum_m \lambda_m \langle\psi|P_m|\psi\rangle \\ &= \langle\psi|\sum_m \lambda_m P_m|\psi\rangle \\ &= \langle\psi|M|\psi\rangle. \end{aligned}$$

Example. Consider an n-qubit system with measurement basis

$$|x\rangle = |x_1\rangle \otimes \cdots \otimes |x_n\rangle$$

where $|x_i\rangle$ is either $|0\rangle$ or $|1\rangle$. Then, a PM is given by

$$M = \sum_x x P_x$$

where

$$P_x = |x\rangle\langle x|.$$

4.1.2 Generalized measurements

A GM is a collection of unitary operators $\{M_m\}$ satisfying

$$\sum_m M_m^\dagger M_m = I.$$

A state $|\psi\rangle$ subjected to a GM $\{M_m\}$ goes into state

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

with probability

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle.$$

It is always true that

$$\sum_m p(m) = 1.$$

In GMs there is a measured outcome but no notion of a "measured value" (i.e., an eigenvalue of an observable). Feasibility of GM's is not stated as an axiom in some (in fact, most elementary) formulations of QM. We will show its feasibility later.

Example. Distinguishability of states. Suppose Alice prepares one of n states $|\psi_i\rangle$, $1 \leq i \leq n$, and gives it to Bob. If $\{|\psi_i\rangle\}$ are orthogonal, Bob can distinguish which state he is given by performing a GM with

$$M_m = |\psi_m\rangle\langle\psi_m|,$$

because

$$p(m) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = \delta_{im}.$$

If states $\{|\psi_i\rangle\}$ are non-orthogonal, there is no GM that can reliably distinguish these states.

4.1.3 POVM measurements

A POVM on a quantum system is a collection $\{E_m\}$ of positive operators satisfying

$$\sum_m E_m = I.$$

When a state $|\psi\rangle$ is subjected to such a POVM, outcome m occurs with probability

$$p(m) = \langle\psi|E_m|\psi\rangle.$$

The state after measurement is not specified. The notion of "repeatability" is dropped in POVMs.

Example. Distinguishability of states (2). Alice gives Bob a qubit prepared in one of two states

$$|\psi_1\rangle = |0\rangle \;\; |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right).$$

Since the states are non-orthogonal there is no measurement that can distinguish $|\psi_1\rangle$ and $|\psi_2\rangle$. But consider the POVM defined by

$$\begin{aligned}
E_1 &\equiv \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1| \\
E_2 &\equiv \frac{\sqrt{2}}{1+\sqrt{2}}\frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2} \\
E_3 &\equiv I - E_1 - E_2
\end{aligned}$$

Outcome E_1 never occurs when the state is $|\psi_1\rangle$. So, if E_2 is measured, the state must be $|\psi_2\rangle$. Likewise, if E_2 is measured, the state must be $|\psi_1\rangle$. If E_3 is measured, no certain conclusion can be drawn. Notice that no mis-identification is made.

4.1.4 Equivalence of POVM and GM

Given a POVM $\{E_m\}$, define

$$M_m = \sqrt{E_m}.$$

Then,

$$\sum_m M_m^\dagger M_m = \sum_m E_m = I,$$

and $\{M_m\}$ is a GM equivalent to $\{E_m\}$.

Conversely, given a GM $\{M_m\}$, define

$$E_m = M_m^\dagger M_m.$$

Then, $\{E_m\}$ is a POVM equivalent to $\{M_m\}$.

4.1.5 GMs as PMs

Let $\{M_m;\ 1 \leq m \leq n\}$ be a GM on a system Q. Let A be an ancilla system of dimension n with basis $\{|m\rangle;\ 1 \leq m \leq n\}$. On system $Q \otimes A$ define a unitary operator whose restriction to states of the form $|\psi\rangle|1\rangle$ is such that

$$U|\psi\rangle|1\rangle \to \sum_m M_m|\psi\rangle|m\rangle.$$

Also on $Q \otimes A$, define a PM $\{P_m \equiv I_Q \otimes |m\rangle\langle m|;\ 1 \leq m \leq M\}$. Now, the GM $\{M_m\}$ on a state $|\psi\rangle$ of Q is equivalent to the PM

$$\sum_m m P_m$$

on state $U|\psi\rangle|1\rangle$ of $Q \otimes A$ in the sense that the same outcomes are obtained with the same probabilities. In particular, on the product system one obtains the measurement outcome

$$\frac{P_m U|\psi\rangle|1\rangle}{\sqrt{\langle\psi|U^\dagger P_m U|\psi\rangle}} = \frac{M_m|\psi\rangle|m\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

with probability

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle.$$

This measurement outcome is an unentangled state and the subsystem Q is in state

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}.$$

4.1.6 Superdense coding

As an application of quantum measurement ideas, we will discuss superdense coding which is an example of quantum communications. In this, Alice conveys to Bob two classical bits by sending one qubit. Initially, Alice and Bob share a pair of qubits in an entangled state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Depending on the message, Alice applies the operator in the table to her qubit (qubit one) and sends it to Bob.

Alice's message	Alice applies operator	Bob obtains state
00	I	$\|00\rangle + \|11\rangle$
01	Z	$\|00\rangle - \|11\rangle$
10	X	$\|10\rangle + \|01\rangle$
11	iY	$\|01\rangle - \|10\rangle$

(For notational convenience, we omitted the factor of $1/\sqrt{2}$ from the states in the table.) The states Bob may obtain are orthogonal (Bell states) and can be reliably distinguished.