5.2 Von Neumann Entropy

The von Neumann entropy of a quantum system in state ρ is defined as

$$S(\rho) = -\mathrm{tr}(\rho \log \rho).$$

In terms of the eigenvalues $\{\lambda_i\}$ of ρ, the von Neumann entropy is given by

$$S(\rho) = \sum_i -\lambda_i \log \lambda_i.$$

Since $\mathrm{tr}(\rho) = \sum_i \lambda_i = 1$, the eigenvalues of ρ form a probability distribution. Hence, the von Neumann entropy of ρ equals the Shannon entropy of its eigenvalues.

For any two density operators ρ and σ on a given state space, the *relative entropy* is defined by

$$S(\rho||\sigma) = \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma)$$

This is the analogue of Shannon relative entropy $D(p||q)$ between two probability distributions p and q on a common set. A basic property of quantum relative entropy is non-negativity, known as Klein's inequality,

$$S(\rho||\sigma) \geq 0,$$

with equality iff $\rho = \sigma$. For a proof, see Nielsen and Chuang.

Some properties of von Neumann entropy are as follows.

1. For any density matrix of size d, $0 \leq S(\rho) \leq \log(d)$ with $S(\rho) = 0$ iff ρ is a pure state and $S(\rho) = \log(d)$ iff ρ is an equally weighted mixture of d orthogonal states, i.e., $\rho = (1/d)I$. This property is immediate from the definition of von Neumann entropy in terms of the Shannon entropy of the eigenvalues of the density matrix.

2. The entropy is unchanged under unitary transformations,

$$S(U\rho U^\dagger) = S(\rho),$$

   since the eigenvalues of ρ are unchanged under such a transformation.

3. For a composite system AB in a pure state ρ^{AB}, $S(\rho^A) = S(\rho^B)$. This is by Schmidt decomposition which says that the eigenvalues of ρ^A and ρ^B form identical sets.

4. Subadditivity. For a joint system AB in state ρ^{AB},

$$S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B).$$

This is proved by noting that

$$S(\rho^{AB}||\rho^A \otimes \rho^B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}),$$

and using Klein's inequality.

5. Concavity. For $\{p_i\}$ a probability distribution,

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i).$$

To prove this consider a composite system in state

$$\rho^{AB} = \sum_i p_i \rho_i \otimes |i\rangle\langle i|$$

where $\{|i\rangle\}$ is an orthonormal basis for subsystem B. The eigenvalues of ρ^{AB} are of the form $p_i \lambda_i^j$ where λ_i^j is an eigenvalue of ρ_i, so

$$S(\rho^{AB}) = H(p_i) + \sum_i p_i S(\rho_i).$$

On the other hand,

$$\rho^A = \sum_i p_i \rho_i, \quad \rho^B = \sum_i p_i |i\rangle\langle i|;$$

so,

$$S(\rho^A) = S(\sum_i p_i \rho_i), \quad S(\rho^B) = H(p_i).$$

The result now follows by subadditivity. Equality holds iff all ρ_i for which $p_i > 0$ are identical.

6. Measurements. Suppose we perform a projective measurement $M = \sum_i \lambda_i P_i$ on a state ρ to obtain outcome λ_i with probability $p_i = \mathrm{tr}(P_i \rho)$. Then, the Shannon entropy of the measurement outcomes is an upperbound on the entropy of ρ,

$$H(p_i) \geq S(\rho).$$

See Nielsen and Chuang p. 515 for a proof.

7. State preparation. If a pure state is chosen randomly from the ensemble $\{|\psi_x\rangle, p_x\}$ so as to prepare the density

$$\rho = \sum_x p_x |\psi_x\rangle\langle\psi_x|,$$

then

$$H(p_x) \geq S(\rho).$$

8. Strong subadditivity. For a tripartite system,

$$S(\rho^{ABC}) + S(\rho^B) \leq S(\rho^{AB}) + S(\rho^{BC}).$$

No simple proof is known for this result. See Nielsen and Chuang for a proof.

9. Araki-Lieb inequality. For a bipartite system,

$$S(\rho^{AB}) \geq |S(\rho^A) - S(\rho^B)|.$$

(Note that the inequality $S(\rho^{AB}) \geq S(\rho^A)$ is false for von Neumann entropy.)