Spring 2002
PHYS/EEE 532 Homework Set 3
Due: 5 April 2002

1. Let x be an irrational number number with an infinite continued fraction expansion $x = [a_0, a_1, \ldots]$. Let $C_k = p_k/q_k = [a_0, \ldots, a_k]$, $k \geq 0$ be the kth convergent.

   (a) Show directly that $C_{2k} \leq x \leq C_{2k+1}$, $k \geq 0$.

   (b) Show by induction that for $k \geq 1$

   $$\begin{aligned} p_{k+1} &= a_{k+1}p_k + p_{k-1} \\ q_{k+1} &= a_{k+1}q_k + q_{k-1} \end{aligned}$$

   with initial values

   $$p_0 = a_0, \quad p_1 = a_1 a_0 + 1,$$
   $$q_0 = 1, \quad q_1 = a_1.$$

   (c) Show that

   $$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}, \quad k \geq 1$$

   and

   $$p_k q_{k-2} - q_k p_{k-2} = (-1)^k a_k, \quad k \geq 2.$$

   Use these to conclude that

   $$C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

   and

   $$C_k - C_{k-2} = \frac{(-1)^k a_k}{q_k q_{k-2}}.$$

   (d) Use the results of the above to show that

   $$C_0 \leq C_2 \leq \cdots C_{2k} \leq x \leq C_{2k+1} \cdots C_3 + C_1.$$

2. The number π has a decimal expansion $\pi = 3.14159265358979\ldots$.

   (a) Find the convergents $C_k = p_k/q_k = [a_0, \ldots, a_k]$ to π for $k = 0, 1, 2, 3, 4$.

   (b) Find the best approximation p/q to π with $q \leq 30,000$. (Use the result in the handout.)

   (c) What is the best approximation p/q to π with $q \leq q_k$, for any $k \geq 0$? (Use the same result.)

3. Use Euclid's algorithm (see p.628 of Nielsen and Chuang) to find $\gcd(693, 147)$. Show the steps.

4. Compute $87^{656} \pmod{119}$ using the algorithm given in Box 5.2 of Nielsen and Chuang. Show the steps.