Spring 2002
PHYS/EEE 532 Homework Set 5
Due: 30 April 2002

1. The *divergence* between two probability distributions $p(x)$, $q(x)$ on a common alphabet is defined as
$$D(p||q) = \sum_x p(x) \log \frac{p(x)}{q(x)}.$$
Prove that $D(p||q) \geq 0$ and find the condition for equality.

2. Prove that the entropy function is concave, i.e., for any two probability distributions $p(x)$ and $q(x)$, and $r(x) = \lambda p(x) + (1 - \lambda)q(x)$, with $0 \leq \lambda \leq 1$,
$$\lambda H(p) + (1 - \lambda)H(q) \leq H(r).$$
Hint: Use the inequality $H(X|Z) \leq H(X)$. Take Z as a binary random variable which takes the value 0 with probability λ and 1 with probability $1 - \lambda$. Let X be a random variable such that the conditional distribution of X given Z is
$$p(x|z) = \begin{cases} p(x), & z = 0, \\ q(x), & z = 1 \end{cases}$$

3. Prove the strong subadditivity: For any X, Y, Z, $H(X,Y,Z) + H(Y) \leq H(X,Y) + H(Y,Z)$ with equality iff $Z \to Y \to X$.

4. The binary erasure channel (BEC) has an input alphabet $\{0,1\}$, output alphabet $\{0,1,?\}$ and transition probabilities
$$p(y|x) = \begin{cases} 1 - \epsilon & \text{if } x = y, \\ \epsilon & \text{if } (x,y) = (0,?) \text{ or } (x,y) = (1,?) \end{cases}$$
Here, "?" signifies erasure and it occurs with probability ϵ. Compute the capacity C of the BEC as a function of ϵ and sketch it for $0 \leq \epsilon \leq 1$.