EE 501 : LINEAR SYSTEM THEORY
EXERCISES ON LINEAR TIME-INVARIANT SYSTEMS

1. Obtain the modal decomposition of the solutions of $\dot{x} = Ax$, for $x_0 = e_i, i = 1, 2, 3, 4$, where

$$A = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

2. Given

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 1 & 1 & \lambda \end{bmatrix}$$

   (a) Find the characteristic and minimal polynomials, a modal matrix and the Jordan form of A.

   (b) Find a *nonzero* initial condition x_o such that the solution $\phi(t, 0, x_o)$ of $\dot{x}(t) = Ax(t), x(0) = x_o$ is orthogonal to $c = [\,1 \quad 1 \quad 1\,]^T$.

3. Consider a system $\mathcal{S} = (A, b, c^T, 0)$ with a transfer function $h(s) = c^T(sI - A)^{-1}b$. Show that if $z \in \mathcal{C}$ is not an eigenvalue of A then

   (a) $\rho(t, 0, (zI - A)^{-1}b, e^{zt}) = h(z)e^{zt}$.

   (b) $\rho(t, 0, (-1)^k k!(zI - A)^{-1}b, t^k e^{zt}) = \sum_{i=0}^{k} \binom{k}{i} h^{k-i}(z)t^i e^{zt}$

4. Consider the circuit shown.

   a) Find the state equations.

   b) Find the step response $\rho(t, 0^-, \theta, 1(t))$.

5. A linear time–invariant system has the impulse response $h(t) = \delta(t) + e^{-t}1(t)$.

   (a) Find the system function $\hat{h}(\omega)$.

   (b) Find the zero–state response to input $u(t) = e^{j\omega_o t}$.

6. Find the modes of the system $\dot{x} = Ax$, with

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

   and decompose the solution corresponding to $x_o = [\, \alpha \quad \beta \quad \gamma \,]^T$ into its modes.

7. Consider the system $[A, b, c^T]$ with

$$\begin{aligned} A &= \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & -2 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ c^T &= [\, 0 \quad 1 \quad 1 \,] \end{aligned}$$

   (a) Find a modal matrix and the Jordan form of A.

   (b) Find the zero–input solution $s(t, 0, x_o, \theta)$ corresponding to $x_0 = [\, 2 \quad 0 \quad 3 \,]^T$, and decompose into modes.

8. Consider the system $[A, b, c^T]$ with

$$\begin{aligned} A &= \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} \alpha \\ 1 \end{bmatrix} \\ c^T &= [\, 0 \quad 1 \,] \end{aligned}$$

   (a) Find α such that the steady state response to $u(t) = 1(t)$ is $y_{ss}(t) = 1$. Find also an initial state $x(0) = x_o$ such that no transients are observed at the output.

   (b) For the value of α found in (a), find the steady state response to $u(t) = t1(t)$.

9. For the following network, find the steady-state response $y_{ss}(t)$ to the input $u(t) = t1(t)$.

10. Consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{aligned}$$

(a) Find the steady-state response to $u(t) = \cos t, t \geq 0$.

(b) Find an initial state such that no transients are observed at the output for $u(t) = 1,\ t \geq 0$, and calculate the resulting output.

11. For a linear, TI system $\rho(t, 0, \theta, \sin t) = \sin t - \cos t + e^{-t} \cos t$. Find $\rho(t, 0, \theta, \cos t)$.

12. Given a linear differential equation

$$\dot{x} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} x$$

Find the solution corresponding to $x(0) = [\ 10\ \ 5\ \ 2\ ]^T$ and decompose into its modes.

13. It is given that $\phi(t, 0, x_o) = [\cos t + \sin t \quad \cos t + 3 \sin t]^T$ is a solution of $\dot{x}(t) = Ax(t)$.

(a) Find x_o.

(b) Find A.

(c) Find the solution corresponding to $x_o = [1 \quad 2]^T$.

14. Consider a system $\dot{x}(t) = Ax(t)$, $y(t) = c^T x(t)$, where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 1 \\ -2 & 0 & -2 \end{bmatrix} \quad , \quad c^T = [\, 0 \quad 0 \quad 1 \,]$$

(a) Obtain a modal matrix and the Jordan form of A.

(b) Find the solution corresponding to $x(0) = [\ 0 \quad 0 \quad 1\ ]^T$ and decompose into modes.

(c) Find an initial state $x(0) = x_o$ such that $y(t) = 0, t \geq 0$.

15. Consider a discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + bu(k) \\ y(k) &= c^T x(k) \end{aligned}$$

where all eigenvalues of A have magnitudes less than unity (i.e., the system is stable). Let $u(k) = \alpha^k, k \geq 0$, where α is not an eigenvalue of A. Characterize an initial state $x(0)$ such that no transients are observed at the output. Find also an expression for the corresponding output.

16. Show that a stable SISO system with a transfer function $h(s) = n(s)/d(s)$ can track an input of the form $u(t) = t^k$ without any steady-state error if an only if

$$\lim_{s \to 0} \frac{d(s) - n(s)}{s^k d(s)} = 0$$

17. Consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= [\ \alpha \quad 0 \quad 1\ ]x \end{aligned}$$

(a) Find α such that $y_{ss}(t) = 1$ for $u(t) = 1, t \geq 0$.

(b) For the value of α found above, calculate $y_{ss}(t)$ for $u(t) = t, t \geq 0$.

(c) Repeat (b) for $u(t) = \cos \omega t, t \geq 0$. Does there exist an ω for which $y_{ss}(t) = 0$?

18. Obtain the modal decomposition of the solution of

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} x$$

corresponding to

(a) $x_0 = [\, 0 \ -1 \ \ 3 \,]^T$

(b) $x_0 = [\, 1 \ -3 \ \ 2 \,]^T$

19. Consider a discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + bu(k) \\ y(k) &= c^T x(k) \end{aligned}$$

where all eigenvalues of A have magnitudes less than unity (i.e., the system is stable). Let $u(k) = \alpha^k, k \geq 0$, where α is not an eigenvalue of A. Characterize an initial state $x(0)$ such that no transients are observed at the output. Find also an expression for the corresponding output.

20. Given a system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u \\ y &= [\, 0 \ \ 0 \ \ 1 \,]\, x + u \end{aligned}$$

(a) Find the transfer function.

(b) Find the steady-state response to $u(t) = \cos \omega t$. For what values of ω, $y_{ss}(t) = 0$?