RECURSIVE MEDIAN FILTERING WITH PARTIAL REPLACES

Adrian Burian and Pauli Kuosmanen

Signal Processing Laboratory, Digital Media Institute
Tampere University of Technology
PO Box 553, FIN-33101, Tampere, Finland
E-mail: burian, pqo @cs.tut.fi

ABSTRACT

The median filter is a special case of nonlinear filter used for smoothing signals. Since the output of the median filter is always one of the input samples, it is conceivable that certain signals could pass through the median filter unaltered. These signals, invariant to further passes of the median filter, define the signature of a filter and are referred to as root signals. This represents the convergence property of median filters. The convergence behavior of different schemes of recursive median filters, and algorithms for image processing using these filters will be studied.

1. INTRODUCTION

Since Tuckey suggested the standard median filter for smoothing statistical time series [1], this concept has been widely studied. By repeating the median filtering, the root signal, which is invariant to further filtering, is found. The existence of root signals is a fundamental property of median filter, used in characterizing these nonlinear filters.

Frequency analysis and impulse response have no meaning in median and recursive median filtering: the impulse response of a recursive median filter is zero. As a result, new tools had to be developed to analyze and characterize the behavior of these nonlinear filters, deterministically and statistically [2], [3], [4]. By associating the nonlinear operation of median filtering with a linear cost function, in [5] was shown that median filtering is an optimization process in which a two-term cost function is minimized.

To compute the output of a 1-D median filter, an odd number of sample values are ranked, and the median value is used as the filter output. If the filter's window length is $N = 2k+1$, the filtering procedure is given by:

$$Y(n) = \text{med}[X(n-k), ..., X(n), ..., X(n+k)], \tag{1}$$

where X(n) and Y(n) are the input and the output sequences, respectively. It is reasonable to assume that the signal is of finite length, consisting of samples from $X(0)$ to $X(L-1)$. To be able to filter the outmost input sample, when parts of the filter's window fall outside the input signal, the input signal is appended to the required size by replicating the outmost input sample as many times as needed. This is the nonrecursive median filter. If the point X(n) is replaced with the output of the median filter before shifting the window to the next position, we obtain the recursive median filter. The output of the recursive filter is given by:

$$Y(n) = \text{med}[Y(n-k), ..., Y(n-1), X(n), ..., X(n+k)]. \tag{2}$$

Recursive median filters use also previously filtered values as their inputs. In this case the filtering operation is performed 'in-place', so that the output of the filter replaces the old input value, before the filter window is moved to the next position. With the same amount of operations recursive filters can usually provide better smoothing capability than nonrecursive filters, at the expense of increased distortion. Recursive median filters have stronger noise attenuation capabilities than their nonrecursive version, and a faster convergence. In [6] simple proofs of the convergence properties of median filters and the idempotent property (reduction to a root after one pass) of recursive median filters are given.

An upper bound of the number of filter passes for median filtering a finite length signal to a root is $(L-2)/2$, where L is the length of the input sequence. This bound is independent of the window width of the filter. A more tight bound is $3(L-2)/[2(k+2)]$, where $N = 2k+1$ is the filter's window width.

For image processing applications, two-dimensional median filters have been used with success. In [7] a new approach for designing the recursive median filter for image processing applications was introduced. The original signal replaces the output of the previous pass in the middle of the operation window. The convergence of this improved recursive median filter within a finite number of iterations was proven. This new scheme of recursive median filter provides an improved MSE performance over the standard recursive median filter. In this paper we further investigate the possibility of using partial replaces of the old input value at the filter's output, before moving the window to the next position. We show that even better MSE performances could be attained by different recursive median filtering schemes. Proofs of the convergence of these recursive median filtering schemes are also given.

2. CENTER WEIGHTED MEDIAN FILTERS

An immediate generalization of the median filter and a major subclass of stack filters are the weighted median (WM) filters [6], [8]. The standard median filter has a better noise attenuation than any WM filter, regardless of the noise distribution. But, in order to preserve small details, WM filters can be the solution.

The output of the WM filter of window size $N = 2k+1$ associated with the integer weights $w_{-k}, ...w_k$ is given by:

$$Y(n) = \text{med}[w_{-k}\Diamond X(n-k), \cdots, w_0\Diamond X(0), \cdots, w_k\Diamond X(n+k)], \quad (3)$$

where the symbol $\Diamond$ is used to denote duplication, i.e.,

$$n\Diamond x = \underbrace{x, \cdots, x}_{n \text{ times}}. \quad (4)$$

Center weighted median (CWM) filters are a subclass of WM filters which combines the simplicity of median filters with some of the design freedom of WM filters. For these filters only the center sample in the window has a weight larger than one. All other weights are equal to one. The CWM filters are the simplest WM filters and the easiest to be designed and implemented. A CWM filter of window width $N = 2k+1$ is defined as:

$$Y(n)^p = \text{med}[X(n-k), \cdots, p\Diamond X(n), \cdots, X(n+k)]. \quad (5)$$

After the center is weighted, the filter is effectively $2k+p$ long, with $p = 2m-1$ smaller than or equal to *k* (otherwise the filter would be reduced to the identity filter). Different values of p produce different CWM filters. When $p = m = 1$ the median filter is obtained, which has the convergence property. When $m = k$, it has been shown that when $k > 1$, the resulting 1-D filter is idempotent.

A CWM filter is completely specified by two parameters: the window size and the center weight. In general, the longer the window size of a CWM filter, the better noise attenuation ability the filter has. CWM filter can be designed to possess good noise attenuation and preserve small details.

In contrast to recursive median filters, which are idempotent, the recursive WM filters usually are not. All the recursive CWM filters corresponding to a WM filter make an arbitrary input signal to converge to a root signal.

3. THRESHOLD DECOMPOSITION

In [9] a powerful tool called threshold decomposition for analyzing rank order based filters was introduced. Using this technique, the analysis of these filters is reduced to studying their effects on binary signals. The importance of the threshold decomposition arises from the fact that binary signals are much easier to analyze than multi-valued signals.

Threshold decomposition of a signal vector $\{X(n)\}$ *M*-valued, where the samples are integer-valued, $0 \leq X(i) < M; 0 \leq i < L$ means decomposing it into *M*-1 binary signal vectors $X^1(n), X^2(n), ..., X^{M-1}(n)$, according to the following rule:

$$X^m = T^m(X(n)) = \begin{cases} 1 & \text{if } X(n) \geq m \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

This thresholding scheme can be applied to any signal that is quantized to a finite number of arbitrary signals. The original multi-valued signal samples X(n) can be reconstructed from the threshold levels by adding them:

$$X(n) = \sum_{m=1}^{M-1} X^m(n). \quad (7)$$

Applying a recursive median filter to an M-valued signal is equivalent to decomposition the signal to M-1 binary threshold signals, filtering each binary signal separately with the corresponding binary recursive median filter, and then reversing the decomposition.

The binary sequence {0, 1} of $\{\ X^m(n)\ \}$ is transferred into {-1, 1} binary sequence of $\{\ Z^m(n)\ \}$ by $Z^m(n) = 2X^m(n) - 1$. For the recursive median filtering of a binary sequence $\{Z^m(n)\}$, the output of the filter is given by:

$$O^m(n) = \begin{cases} +1 & \text{if } S(n) \geq 0 \\ -1 & \text{otherwise,} \end{cases} \quad (8)$$

where

$$S(n) = \sum_{j=-N}^{N} Z^m(n+j). \quad (9)$$

4. DIFFERENT RECURSIVE MEDIAN FILTERING SCHEMES

It is known that in the case of 2-D signals the recursive median filters are not necessarily idempotent [3]. Thus, in order to find the root signal, it is necessary to apply the recursive median filter iteratively. For the recursive median filter, at each iteration for every point of the image we have to compute the output of the filter:

$$O_t^m(n) = \text{med}[O_{t-1}^m(n-k), ...O_{t-1}^m(n), ..., O_{t-1}^m(n+k)], \quad (10)$$

where the subscript t represents the iteration index. The recursive median filtering is a sequential process and the noise influence at his output will be accumulated. To alleviate such an undesirable effect it may be useful to encourage the filter output to resemble the original signal. The recursive median filtering is an optimization operation in which the output of the filter is always set to the minimum of a cost function of the output state of the filter [5]. The first recursive median filtering scheme is obtained when the output of the filter at each iteration for every point of the image is computed with:

$$O_t^m(n) = \text{med}[O_{t-1}^m(n-k), \cdots, O_{t-1}^m(n-1), \\ Z^m(n), O_{t-1}^m(n+1), \cdots, O_{t-1}^m(n+k)]. \quad (11)$$

So, in this case, instead of using the output of the previous pass, the value from the middle of the window of the filter is replaced by the original signal. The obtained filter is convergent to a root signal within a finite number of iterations [7].

This approach is extended by choosing different positions and several points inside the filter's window to be replaced by the original signal, instead of the outputs of the previous passes. At each step of the filtering process, the following function is minimized:

$$E(O^m(n)) = -\sum_{j1} O^m(n)O^m(n+j1) \\ -\sum_{j2} O^m(n)Z^m(n+j2), \quad (12)$$

where $j1 + j2 = 2k + 1$ and $j2$ represents the number of the points from the original signal that replace the values from the filter's window. The first term gives a measure of the smoothness of the filtering process, and the second one measures the discrepancy between the filter output and the original signal.

In the case of the recursive median filter, each point of the signal is sequentially visited and the output is updated before moving to the next position. For the whole process, the following function will be minimized:

$$E = -\sum_{n=1}^{L}\sum_{j1} O^m(n)O^m(n+j1) \\ -\sum_{n=1}^{L}\sum_{j2} O^m(n)Z^m(n+j2). \quad (13)$$

Because the process is sequential and at any time only one output is changed, the changes of the function E from one 'global' step to another are given by:

$$\Delta E = \Delta O^m(n)S(n). \quad (14)$$

The value of ΔE is less than or equal to zero (from (8)), so after a finite number of iterations, E will reach its minimum and the filtered signal will be reduced to a root.

Notice that $j2$ cannot take values greater than $2k$; otherwise, the recursive process became meaningless. If $j2 = 1$ the only replacement takes place in the middle of the window of the filter. The equation (11) is obtained.

For the other extreme case $j2 = 2k$, all values except the middle of the window of the filter are replaced with the original signal. At the second stage, the value for $\Delta O_i(n)$ will be zero, even in the 2-D case, so the obtained filter structure is idempotent. This appears because at the second stage we have:

$$O_{t+1}(n) = O_{t=0}(n) = \text{med}[Z(n-k), \cdots, Z(n+k)]. \quad (15)$$

In this case, the filtering scheme is given by:

$$O_t^m(n) = \text{med}[Z^m(n-k), \cdots, Z^m(n-1), \\ O_{t-1}^m(n), Z^m(n+1), \cdots, Z^m(n+k)]. \quad (16)$$

5. EXPERIMENTAL RESULTS

In order to objectively evaluate the performances of these new filtering schemes, we have used the Lena image corrupted by impulsive noise. The image considered contained 256x256 pixel values with 8 bits resolution per pixel. In all cases, a 3x3 window was used and the threshold decomposition technique was applied. The scanning order was line by line. The results were similar when a column by column scanning was used.

For the simulations we have considered five different filtering schemes for recursive median and CWM filters. In all plots with continuous line without markers the traditional recursive median and CWM filters were represented. The filtering scheme given by (11) is marked with a circle and the one given by (16) with a star. An intermediate simulation, which replaces all the corners of the window of the filter with the original signal is marked with a cross. In order to eliminate as much as possible the discrepancy between the filter output and the original signal, the positions of the replacements were varied during the filtering process, depending on the actual local value of the corruption. The results for this simulation are marked with a triangle. The best results were obtained in this case.

Figure 1 presents the computed Mean Square Error (MSE) and Mean Absolute Error (MAE) for different recursive median filtering schemes. In Figure 2 the computed MSE and MAE for the same recursive median filtering schemes, but with a central weight of value $p = 3$ are presented.

6. CONCLUSIONS

In this paper some recursive median filtering schemes for image processing were introduced. The convergence properties of these filtering schemes were studied. The results

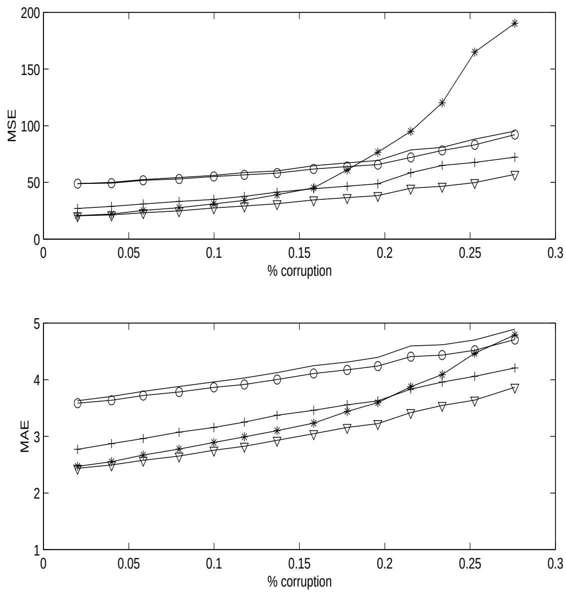

Figure 1: Performance evaluation results.

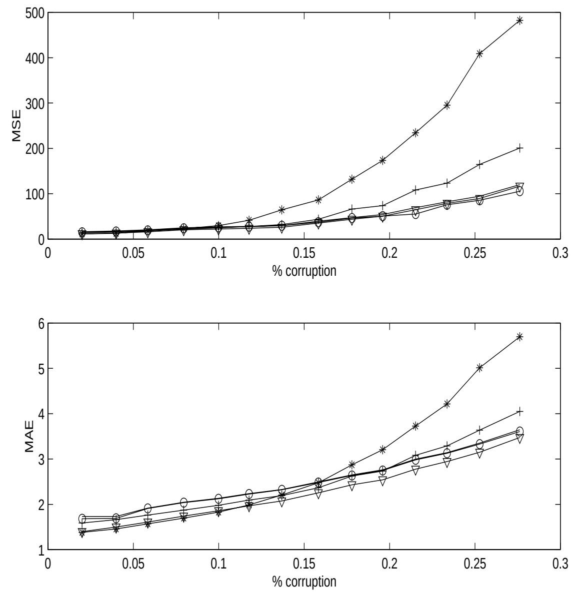

Figure 2: Performance evaluation results for CWM.

of the simulations illustrate the improving of the MSE and MAE performances over the traditional methods.

7. REFERENCES

[1] J.W. Tukey, "Nonlinear (NonSuperposable) Methods for Smoothing Data", *Cong. Rec. EASCON'74*, pp. 673, 1974.

[2] J.P. Fitch, E.J. Coyle, N.L. Gallagher Jr., "Root Properties and Convergence Rates of Median Filters", *IEEE Trans. on Acoustic, Speech and Signal Processing*, vol. ASSP-33, no. 1, pp. 230-239, Feb. 1985.

[3] T.A. Nodes, N.L. Gallagher Jr., "Two-Dimensional Root Structures and Convergence Properties of the Separable Median Filter", *IEEE Trans. on Acoustic, Speech and Signal Processing*, vol. ASSP-31, no. 6, pp. 1350-1365, Dec. 1983.

[4] T.A. Nodes, N.L. Gallagher Jr., "Median Filters: Some Modifications and Their Properties", *IEEE Trans. on Acoustic, Speech and Signal Processing*, vol. ASSP-30, no. 5, pp. 739-746, Oct. 1982.

[5] G. Qiu, "Functional Optimization Properties of Median Filtering", *IEEE Signal Processing Letters*, vol. 1, no. 4, pp. 64-66, April 1994.

[6] B. Zeng, "Convergence Properties of Median and Weighted Median Filters", *IEEE Trans. on Signal Processing*, vol. 42, no. 12, pp. 3515-3519, Dec. 1994.

[7] G. Qiu, "An Improved Recursive Median Filtering Scheme for Image Processing", *IEEE Trans. on Image Processing*, vol. 5, no. 4, pp. 646-648, April 1996.

[8] O. Yli-Harja, J. Astola, Y. Neuvo, "Analysis of the Properties of Median and Weighted Median Filters using Threshold Logic and Stack Filter Representation", *IEEE Trans. on Acoustic, Speech and Signal Processing*, vol. ASSP-39, no. 2, pp. 395-410, Feb. 1991.

[9] J.P. Fitch, E.J. Coyle, N.L. Gallagher Jr., "Median Filtering by Threshold Decomposition", *IEEE Trans. on Acoustic, Speech and Signal Processing*, vol. ASSP-32, no. 6, pp. 1183-1188, Dec. 1984.